# A macroscopic model for a system of swarming agents using curvature control

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**Abstract**

In this paper, we study the macroscopic limit of a new model of collective displacement. The model, called PTWA, is a combination of the Vicsek alignment model [41] and the Persistent Turning Walker (PTW) model of motion by curvature control [21, 24]. The PTW model was designed to fit measured trajectories of individual fish [24]. The PTWA model (Persistent Turning Walker with Alignment) describes the displacements of agents which modify their curvature in order to align with their neighbors. The derivation of its macroscopic limit uses the non-classical notion of generalized collisional invariant introduced in [20]. The macroscopic limit of the PTWA model involves two physical quantities, the density and the mean velocity of individuals. It is a system of hyperbolic type but is non-conservative due to a geometric constraint on the velocity. This system has the same form as the macroscopic limit of the Vicsek model [20] (the 'Vicsek hydrodynamics') but for the expression of the model coefficients. The numerical computations show that the numerical values of the coefficients are very close. The 'Vicsek Hydrodynamic model' appears in this way as a more generic macroscopic model of swarming behavior as originally anticipated.

**Key words:** Individual based model, Fish behavior, Persistent Turning Walker model, Vicsek model, Orientation interaction, Asymptotic analysis, Hydrodynamic limit, Collision invariants.

**AMS Subject classification:** 35Q80, 35L60, 82C22, 82C31, 82C70, 82C80, 92D50.

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# 1 Introduction

Modeling swarming behavior has attracted a lot of attention in the recent years. To model a flock of birds [2], a school of fish [16, 29, 36, 42] or the displacement of ants [14, 30, 40], a key question is to understand how to relate the collective behavior of large groups of agents to simple individual mechanisms [7, 15]. From a mathematical point of view, this question takes the form of the derivation of macroscopic equations from individual based models [3, 4, 13, 20, 23]. This paper is devoted to the derivation of a macroscopic model for a new type of model of collective behavior where agents control their motion by changing the curvature of their trajectory. This model has been shown to provide the best fit of fish trajectories [24].

Among models of collective displacements, the so-called Vicsek model has received a particular attention [18, 41]. This model describes the tendency of individuals to align with their congeners. Many features of this model have been studied such as the existence of a critical point [12, 41], the long time behavior [12, 34] or the derivation of a continuum model [4, 20]. Due to its simplicity, several extensions or modifications of this model have been proposed, such as the Cucker-Smale model [8, 9, 17, 27, 28]. There is also a variety of models which add an attraction and a repulsion rule to the Vicsek model [12, 22]. However, the Vicsek model has been proposed on phenomenological bases. By contrast, the experiments of [24] have shown that the Persistent Turning Walker (PTW) model provides the best fit to individual fish trajectories. In the PTW model, the individual controls its motion by acting on the curvature of its trajectory instead of acting on its velocity. However, in its version of [21, 24], the PTW model only describes the evolution of a single individual. The model does not take into account the interactions between congeners.

In the present work, interactions between individuals are introduced in the PTW model by means of an alignment rule, like in the Vicsek model. The resulting model, called PTWA (Persistent Turning Walker with Alignment) describe how each individual is influenced by the average velocity of its surrounding neighbors. In the framework of the PTW model where individuals control their motion by acting of the curvature of their trajectory, this influence must lead to a modification of this curvature. This contrasts with the Vicsek model, where particles are directly modifying their velocity as a result of the interaction.

The PTWA model is based on the assumption that the subjects use the time derivative of their trajectory curvature (or of their acceleration) as a control variable for planning their movement. Such models are not commonplace in the literature. Their first occurrence is, to the best or our knowledge, in [21, 24]. The present work is the first one in which interaction among the agents is taken into account within this kind of models (see also [25]). We note that [38] introduces the acceleration of neighbors in the rule updating the subjects' velocities in a variant of the Vicsek Individual-Based model [41] but motion planning is eventually made by updating the velocity and not the acceleration.

Once the PTWA model is set up, the main task of the present paper is to derive the macroscopic limit of this new model. This macroscopic limit is intended to provide a simplified description of the system at large scales. The major problem for this derivation is that there is nothing like momentum or energy conservation in the PTWA model. Such conservation laws are the corner stone of the classical theory of macroscopic limits in kinetic theory [11, 19]. Indeed, as a consequence of this absence of conservation, the dimension of the manifold of local equilibria in the PTWA model is larger than the dimension of the space of collisional invariants. Conservation laws are therefore missing for providing a closed set of equations for the macroscopic evolution of the parameters of the local equilibria. To overcome this problem, we use the notion of generalized collisional invariant introduced in [20]. Thanks to this new notion, a closed set of macroscopic equations for the PTWA model can be derived.

The macroscopic model consists of a conservation equation for the local particle density and an evolution equation for the average velocity. The latter is constrained to be of unit norm. The resulting system is a non-conservative hyperbolic which shows similarities but also striking differences to the Euler system of gas dynamics. It has also the same form as the previously derived macroscopic limit of the Vicsek model (also referred to as the 'Vicsek Hydrodynamic model') in [20], but for the expression of the model coefficients. At the end of the paper, we propose a numerical method to compute the generalized collisional invariant out of which the coefficients of the macroscopic model are derived. The similarity between the 'Vicsek hydrodynamics' and the 'PTWA hydrodynamics' can be better understood by considering the relations between the microscopic models. Indeed, the Vicsek model can be seen as a special limit of the PTWA model in a well-suited asymptotic limit. Work is in progress to establish this connexion firmly.

The inclusion of the alignment rule in the PTW model changes drastically the large scale dynamics of the system. Without this alignment rule, the PTW model exhibits a diffusive behavior at large scales [10, 21]. By contrast, when the alignment rule is included, the model becomes of hyperbolic type. Therefore, the local alignment rule added to the PTW model generates convection at the macroscopic scale.

Since the addition of the alignment rule modifies drastically the dynamics of the PTW model, it is also interesting to study the large scale effects of other types of local rules such as attraction-repulsion. The goal is to find a common framework for the large scale dynamics of a large class of swarming models. Currently, there exists a profusion of individual based models, especially for fish behavior (see [36] for a short review). In a macroscopic model, only the gross features of the microscopic model remain. Therefore, the derivation of macroscopic models may be a tool to better capture the common features and differences between these different types of swarming models.

The outline of the paper is as follows: in section 2, we introduce the PTWA model and the main result is stated. Section 3 is devoted to the proof of the derivation of the macroscopic limit of the PTWA model. In section 4, we study some properties of the so-obtained macroscopic model and we numerically estimate the involved coefficients. Finally, in section 5, we draw a conclusion of this work.

# 2 Presentation of the model and main result

## 2.1 The individual based model

The starting point is a model in which alignment interaction between agents is introduced inside the Persistent Turning Walker model (PTW) [21, 24]. The PTW model is a model for individual displacements which has been derived to fit experimentally observed trajectories of fish. It supposes that individuals control their motion by acting of the curvature of their trajectory. To make it a realistic model for collective displacements, the PTW model must be enriched by introducing inter-individual interactions. Indeed, one of the main features of collective motion such as those observed in animal populations (fish schools, mammalian herds, etc.) is the ability of individuals to coordinate with each other. Observations suggest that trend to alignment is an important component of this interaction and leads to a powerful coordination-building mechanism by synchronizing the agent's velocities one to each other. One of the simplest models of alignment interaction is the Vicsek model [41]. This time-discrete model supposes that individuals move at constant speed and align to the average velocity of their neighbors (up to a certain stochastic uncertainty) at each time step. A time-continuous version of this dynamics has been derived in [20].

In order to combine the PTW displacement model and the Vicsek alignment interaction model (in the time-continuous framework of [20]), we propose the following model further referred to ad the PTWA model (PTW model with alignment): among a population of $N$ agents, the motion of the i$^{\text{th}}$ individual is given by

$$\frac{d\mathbf{x}_i}{dt} = c\vec{\tau}(\theta_i), \tag{2.1}$$

$$\frac{d\theta_i}{dt} = c\kappa_i, \tag{2.2}$$

$$d\kappa_i = a(\nu\overline{\kappa}_i - \kappa_i)\,dt + b\,dB_t^i, \tag{2.3}$$

with

$$\overline{\kappa}_i = \vec{\tau}(\theta_i) \times \overline{\Omega}_i \tag{2.4}$$

and

$$\overline{\Omega}_i = \frac{\mathbf{J}_i}{|\mathbf{J}_i|} \quad , \quad \mathbf{J}_i = c \sum_{|\mathbf{x}_i - \mathbf{x}_j| < R} \vec{\tau}(\theta_j), \tag{2.5}$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ is the position of the individual, $\vec{\tau}(\theta_i) = (\cos\theta_i\,, \sin\theta_i)$ is the direction of its velocity vector, with the angle $\theta_i \in (-\pi, \pi]$ measured from the $x_1$ direction, $\kappa_i \in \mathbb{R}$ is the curvature of its trajectory and $B_t^i$ is a standard Brownian motion (with $B_t^i$ independent of $B_t^j$ for $i \neq j$). The magnitude of the velocity is constant and denoted by $c > 0$. The constant $a$ is a relaxation frequency and $b$ quantifies the intensity of the random perturbation of the curvature. The vector $\overline{\Omega}_i$ is the mean direction of the neighbors of the i$^{\text{th}}$ individual (defined as the individuals $j$ which are at a distance less than $R$ from $\mathbf{x}_i$, $R > 0$ being the perception distance of the individuals, supposed given).

The trend to alignment is modeled by the relaxation term of (2.3) (in factor of $a$). It describes the relaxation of the trajectory curvature to the target curvature $\overline{\kappa}_i$. $\overline{\kappa}_i$ is computed by taking the cross product[1] of the direction of the individual $\vec{\tau}(\theta_i)$ and the mean direction of its neighbors $\overline{\Omega}_i$. $\nu\overline{\kappa}_i$ is the trajectory curvature the individual must achieve in order to align to its neighbors. It increases with increasing difference between the individual's velocity and the target velocity. $\nu$ is the typical value of the individuals' trajectory curvature and can be seen as the 'comfort' curvature. The larger $\nu$ is, the faster alignment occurs. The second term of (2.3) (in factor of $b$) is a random term which describes the tendency of individuals to desynchronize to their neighbors in order for instance, to explore their environment. At equilibrium, these two antagonist effects lead to a stationary distribution of curvatures which is the building block of the construction of the macroscopic model.

We illustrate this model in figure 1. In the left figure, a fish is represented turning to the left. However, its neighbors are moving towards the other direction ($\overline{\Omega}$ is pointing to the right). Then the fish is going to adjust its curvature in

[1] For two-dimensional vectors $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, the cross product $\vec{a} \times \vec{b}$ is the scalar $a_1 b_2 - a_2 b_1$.

order to move towards the same direction as $\overline{\Omega}$ (right figure). The adjustment of its curvature requires a certain time of order $1/a$, after which the curvature $\kappa$ is close to $\nu\overline{\kappa}$. Therefore, in this model, there is a time delay between the current acceleration of the fish ($\kappa$) and its desired acceleration ($\nu\overline{\kappa}$). In most models describing animal behavior, the dynamics is inspired by Newton's second law: the acceleration of an individual is equal to a force term which incorporates all information about the environment. In the present model, individuals need a certain time to adjust their acceleration. This rule can be seen as a modification of Newton's second law saying that the force is proportional to the time derivative of the acceleration rather than to the acceleration itself.

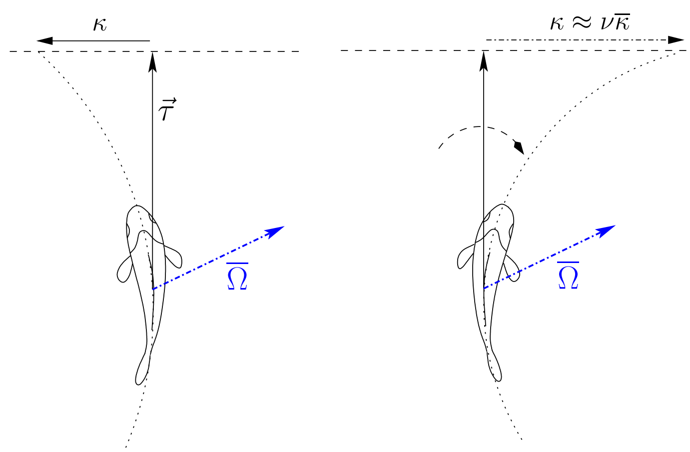


Figure 1: Illustration of the model (2.1)-(2.3). On the left figure, a fish is turning to the left, while its neighbors are moving to the right ($\overline{\Omega}$). After a certain time of order $1/a$, the fish adjusts its curvature in order to align its velocity with $\overline{\Omega}$ (right figure).

Our goal is the study of model (2.1)-(2.3) at large time and space scales. For this purpose, it is convenient to introduce scaled variables. We use $x_0 = \nu^{-1}$ as space unit, $t_0 = (c\nu)^{-1}$ as time unit, $\kappa_0 = x_0^{-1} = \nu$ as curvature unit. We introduce the dimensionless time, space and curvature as $t' = t/t_0$, $\mathbf{x}' = \mathbf{x}/x_0$ and $\kappa' = \kappa/\kappa_0$ and for simplicity we omit the primes in the discussion below. In scaled variables, the PTWA model is given by (for the i$^{\text{th}}$ individual) :

$$\frac{d\mathbf{x}_i}{dt} = \vec{\tau}(\theta_i), \tag{2.6}$$

$$\frac{d\theta_i}{dt} = \kappa_i, \tag{2.7}$$

$$d\kappa_i = \lambda(\overline{\kappa}_i - \kappa_i)\,dt + \sqrt{2}\alpha\,dB_t^i, \tag{2.8}$$

with $\overline{\kappa}_i$ defined by equation (2.4),(2.5) ($c$ being replaced by 1) and $\lambda$, $\alpha$ given by:

$$\lambda = \frac{a}{c\nu} \qquad , \qquad \alpha^2 = \frac{b^2}{2c\nu^3}.$$

## 2.2 Main result

A first step consists in providing a mean-field description of the PTWA dynamics. Introducing the probability density function of fish $f(t, \mathbf{x}, \theta, \kappa)$, we will prove formally that the PTWA model (2.6)-(2.8) leads to the following equation for $f$:

$$\partial_t f + \vec{\tau}(\theta) \cdot \nabla_{\mathbf{x}} f + \kappa \partial_\theta f + \lambda \partial_\kappa \Big[ (\overline{\kappa} - \kappa) f \Big] = \alpha^2 \partial^2_\kappa f, \tag{2.9}$$

with

$$\overline{\kappa} = \vec{\tau}(\theta) \times \overline{\Omega}(\mathbf{x}) \tag{2.10}$$

and

$$\overline{\Omega}(\mathbf{x}) = \frac{\mathbf{J}(\mathbf{x})}{|\mathbf{J}(\mathbf{x})|} \quad , \quad \mathbf{J}(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|<R,\, \theta, \kappa} \vec{\tau}(\theta) f(\mathbf{y}, \theta, \kappa) \, d\mathbf{y} d\theta d\kappa. \tag{2.11}$$

The main concern of this paper is the study of the so-called hydrodynamic limit of the mean-field model (2.9). With this aim, we perform a new rescaling and introduce the macroscopic variables $\tilde{t}$ and $\tilde{\mathbf{x}}$:

$$\tilde{t} = \varepsilon t \qquad , \qquad \tilde{\mathbf{x}} = \varepsilon \mathbf{x}, \tag{2.12}$$

with $\varepsilon > 0$ a small number representing the ratio between the microscopic and the macroscopic time and space scales. In this paper, we give a formal proof that the density distribution of individuals in these new variables $f^\varepsilon(\tilde{t}, \tilde{\mathbf{x}}, \theta, \kappa)$ converges in the limit $\varepsilon \to 0$ to the solutions of a hydrodynamic like model. More precisely, the theorem reads (dropping the tildes for simplicity):

**Theorem 1** *In the limit $\varepsilon \to 0$, the distribution $f^\varepsilon$ converges to an equilibrium:*

$$f^\varepsilon \overset{\varepsilon \to 0}{\rightharpoonup} \rho \mathcal{M}_\Omega(\theta) \mathcal{N}(\kappa)$$

*with $\mathcal{M}_\Omega$ and $\mathcal{N}$ (resp.) a Von Mises distribution and a Gaussian distribution defined at (3.12) and (3.10). The density $\rho = \rho(\mathbf{x}, t)$ and the direction of the flux $\Omega = \Omega(\mathbf{x}, t)$ satisfy the following system:*

$$\begin{aligned} &\partial_t \rho + c_1 \nabla_{\mathbf{x}} \cdot (\rho \Omega) = 0, \\ &\rho \Big( \partial_t \Omega + c_2 (\Omega \cdot \nabla_{\mathbf{x}}) \Omega \Big) + \tfrac{\alpha^2}{\lambda^2} (Id - \Omega \otimes \Omega) \nabla_{\mathbf{x}} \rho = 0, \end{aligned} \tag{2.13}$$

*where $c_1$ and $c_2$ are two positive constants defined later on at (3.29) (3.38).*

The so-obtained macroscopic model (2.13) has the same form as that derived from the Vicsek model [20]. Indeed, the two models only differ by the values of their coefficients. This model is a hyperbolic system which bears some similarities with the Euler system of isothermal compressible gases. There are however some striking differences. First, the convection speed of the density $\rho$ is different from the convection speed of the velocity $\Omega$ ($c_1 \neq c_2$ in general). Moreover, the velocity $\Omega$ is a unit vector and therefore it satisfies the constraint $|\Omega| = 1$. This explains why the pressure term is premultiplied by the matrix $(\mathrm{Id} - \Omega \otimes \Omega)$. This projection matrix guarantees that the resulting vector is orthogonal to $\Omega$. Consequently, the constraint $|\Omega| = 1$ is preserved by the dynamics. However, the projection matrix leads to a non-conservative model which cannot be put in conservative form. This intrinsic non-conservation feature is the macroscopic counterpart of the lack of momentum conservation at the microscopic level (see below).

The modification of the PTW model leading to the PTWA model has drastically changed the nature of the macroscopic model. Indeed, the macroscopic limit of the PTW model without the incorporation of the interactions is of diffusive nature [10, 21]. By contrast, that of the PTWA model is of hyperbolic type. Indeed, the scaling (2.12) is of hydrodynamic type, the macroscopic time and space scales being of the same order of magnitude. By contrast, a diffusive scaling would have required $\tilde{t} = \varepsilon^2 t$ instead (see [10, 21]).

The similarity with the 'Vicsek Hydrodynamics' also confirms that the chosen interaction rule generates alignment since the PTWA model has the same macroscopic limit as the Vicsek model. At the microscopic scale, the PTWA and Vicsek models look rather different, whereas, at the macroscopic scale, they are similar. This is an example of how the derivation of macroscopic model can be used as a tool to reduce and unify different types of swarming models in classes leading to similar macroscopic models.

# 3 Derivation of a macroscopic model

## 3.1 Mean field equation

In this section, we briefly summarize the first step of the derivation of the macroscopic model, namely the derivation of the intermediate mean-field equation (2.9) from the particle dynamics (2.6)-(2.8). In order to derive this mean field equation, we start by looking at the system without the white noise $dB_i^t$ for a large (but fixed) number of individuals $N$. In this case, the system reduces to a coupled system of ordinary differential equations. We denote by $\{X_i(t), \Theta_i(t), K_i(t)\}_{i=1\dots N}$ the solution of this system on a given time interval. Following the standard methodology (see e.g. the text book [37]), we introduce the so-called empirical distribution $f^N$

given by:

$$f^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_i(t)} \otimes \delta_{\Theta_i(t)} \otimes \delta_{K_i(t)}. \tag{3.1}$$

We can easily check that this density distribution satisfies the following equation (weakly):

$$\partial_t f^N + \vec{\tau}(\theta)\cdot\nabla_{\mathbf{x}} f^N + \kappa\partial_\theta f^N + \lambda\partial_\kappa\Big[(\overline{\kappa}^N - \kappa) f^N\Big] = 0,$$

with

$$\overline{\kappa}^N = \vec{\tau}(\theta)\times\overline{\Omega}^N(\mathbf{x})$$

and

$$\overline{\Omega}^N(\mathbf{x}) = \frac{\mathbf{J}^N(\mathbf{x})}{|\mathbf{J}^N(\mathbf{x})|} \quad , \quad \mathbf{J}^N(\mathbf{x}) = \sum_{j,\,|\mathbf{x}-\mathbf{x}_j|<R} \vec{\tau}(\theta_j).$$

The term $J^N$ can be expressed using the empirical distribution $f^N$:

$$\mathbf{J}^N(\mathbf{x}) = N\int_{|\mathbf{x}-\mathbf{y}|<R,\,\theta,\kappa} \vec{\tau}(\theta) f^N(\mathbf{y},\theta,\kappa)\, d\mathbf{y} d\theta d\kappa.$$

Then it is clear that the formal limit $N\to\infty$ of $f^N$ satisfies the following equation:

$$\partial_t f + \vec{\tau}(\theta)\cdot\nabla_{\mathbf{x}} f + \kappa\partial_\theta f + \lambda\partial_\kappa\Big[(\overline{\kappa} - \kappa) f\Big] = 0,$$

with $\overline{\kappa}$ given by (2.10),(2.11).

When the white noise is added, the situation is more complicated. At the particle level (2.6)-(2.8), the system becomes a coupled system of stochastic differential equations. This implies that the empirical distribution $f^N$ given by (3.1) becomes a stochastic measure. In this case, formal considerations suggest that, in the limit $N\to\infty$, the distribution function $f$ satisfies the following Fokker-Planck equation (2.9) with $\overline{\kappa}$ given by (2.10),(2.11). For related questions, we refer the reader to [5, 8, 39].

## 3.2 Hydrodynamic scaling

In order to derive a macroscopic equation from the mean-field equation (2.9)-(2.11), we use the hydrodynamic scaling. With this aim, we introduce the macroscopic variables $\tilde{t}$ and $\tilde{\mathbf{x}}$ defined by (2.12). In the rescaled variables, the distribution function (denoted by $f^\varepsilon$) is given by $f^\varepsilon(\tilde{t},\tilde{\mathbf{x}},\theta,\kappa) = \frac{1}{\varepsilon^2} f(t,\mathbf{x},\theta,\kappa)$. After omitting the tildes, it satisfies the following equation:

$$\varepsilon(\partial_t f^\varepsilon + \vec{\tau}(\theta)\cdot\nabla_{\mathbf{x}} f^\varepsilon) + \kappa\partial_\theta f^\varepsilon + \lambda\partial_\kappa\Big[(\overline{\kappa}^\varepsilon - \kappa) f^\varepsilon\Big] = \alpha^2\partial_\kappa^2 f^\varepsilon, \tag{3.2}$$

with

$$\overline{\kappa}^\varepsilon = \vec{\tau}(\theta)\times\overline{\Omega}^\varepsilon(\mathbf{x})$$

and

$$\overline{\Omega}^{\varepsilon}(\mathbf{x}) = \frac{\mathbf{J}^{\varepsilon}(\mathbf{x})}{|\mathbf{J}^{\varepsilon}(\mathbf{x})|} \quad , \quad \mathbf{J}^{\varepsilon}(\mathbf{x}) = \int_{|\mathbf{x}-\mathbf{y}|<\varepsilon R,\, \theta, \kappa} \vec{\tau}(\theta) f^{\varepsilon}(\mathbf{y}, \theta, \kappa)\, d\mathbf{y} d\theta d\kappa. \tag{3.3}$$

We note that the expression (3.3) of $\mathbf{J}^{\varepsilon}$ supposes that the radius of interaction between the individuals is tied to the microscopic scale. This assumption translates the fact that in most biological system, each individual has only access to information about its close neighborhood. Thanks to this assumption, we can replace the expression of $\overline{\Omega}^{\varepsilon}$ by a local expression. This is precisely stated in the following lemma, the proof of which is obvious and omitted.

**Lemma 3.1** *We have the expansion:*

$$\overline{\Omega}^{\varepsilon} = \Omega_{f^{\varepsilon}} + O(\varepsilon^2),$$

*where*

$$\Omega_{f^{\varepsilon}}(\mathbf{x}) = \frac{\mathbf{j}^{\varepsilon}(\mathbf{x})}{|\mathbf{j}^{\varepsilon}(\mathbf{x})|} \quad \textit{and} \quad \mathbf{j}^{\varepsilon}(\mathbf{x}) = \int_{\theta, \kappa} \vec{\tau}(\theta) f^{\varepsilon}(\mathbf{x}, \theta, \kappa)\, d\theta d\kappa.$$

Finally, we can simplify (3.2) using the equality:

$$\vec{\tau}(\theta) \times \Omega = \sin(\overline{\theta} - \theta)$$

with $\overline{\theta}$ such that:

$$\vec{\tau}(\overline{\theta}) = \Omega_{f^{\varepsilon}}.$$

With these notations, equation (3.2) can be written as:

$$\varepsilon\Big(\partial_t f^{\varepsilon} + \vec{\tau}(\theta) \cdot \nabla_{\mathbf{x}} f^{\varepsilon}\Big) = Q(f^{\varepsilon}) + O(\varepsilon^2) \tag{3.4}$$

with the operator $Q$ (below referred to as the 'collision operator') defined by:

$$Q(f) = -\kappa \partial_{\theta} f - \lambda \sin(\overline{\theta} - \theta)\, \partial_{\kappa} f + \lambda \partial_{\kappa}(\kappa f) + \alpha^2 \partial_{\kappa}^2 f, \tag{3.5}$$

where $\vec{\tau}(\overline{\theta}) = \Omega_f(\mathbf{x})$ defined as:

$$\Omega_f(\mathbf{x}) = \frac{\mathbf{j}(\mathbf{x})}{|\mathbf{j}(\mathbf{x})|} \quad , \quad \mathbf{j}(\mathbf{x}) = \int_{\theta, \kappa} \vec{\tau}(\theta) f(\mathbf{x}, \theta, \kappa)\, d\theta d\kappa. \tag{3.6}$$

In the sequel, we will drop the $O(\varepsilon^2)$ remainder which has no influence in the final result.

## 3.3 Study of the collision operator

### 3.3.1 Equilibria

In order to study the limit $\varepsilon \to 0$ of the solution $f^\varepsilon$ of (3.4), we first have to determine the equilibria of the operator $Q$ defined by (3.5). With this aim, we notice that $Q$ can be decomposed as a sum of a formally skew-adjoint operator and of a formally self-adjoint operator. For the skew-adjoint part, we introduce the function:

$$H(\theta, \kappa) = -\lambda \cos\theta + \frac{\kappa^2}{2}$$

and we adopt the convention that for any function $h(\theta, \kappa)$:

$$h_\Omega(\theta, \kappa) = h_{\overline{\theta}}(\theta, \kappa) = h(\theta - \overline{\theta}, \kappa), \tag{3.7}$$

with $\vec{\tau}(\overline{\theta}) = \Omega$. Using these notations, for any smooth function $f$, the skew-adjoint part of $Q$ can be written as:

$$-\kappa\partial_\theta f - \lambda \sin(\overline{\theta} - \theta)\, \partial_\kappa f = \partial_\theta H_{\overline{\theta}}\, \partial_\kappa f - \partial_\kappa H_{\overline{\theta}}\, \partial_\theta f = \{H_{\overline{\theta}},\, f\}_{(\theta,\kappa)}, \tag{3.8}$$

using the Poisson Bracket formalism $\{\cdot,\cdot\}_{(\theta,\kappa)}$ in the $(\theta, \kappa)$ space. Therefore, any function of the form $g(H_{\overline{\theta}})$ satisfies $\{H_{\overline{\theta}},\, g(H_{\overline{\theta}})\} = 0$. On the other hand, the self-adjoint part of $Q$ satisfies:

$$\lambda\partial_\kappa(\kappa f) + \alpha^2\partial_\kappa^2 f = \alpha^2\partial_\kappa\left(\mathcal{N}\partial_\kappa\left(\frac{f}{\mathcal{N}}\right)\right), \tag{3.9}$$

with $\mathcal{N}$ the Gaussian distribution with zero mean and variance $\alpha^2/\lambda$:

$$\mathcal{N}(\kappa) = \sqrt{\frac{\lambda}{2\pi\alpha^2}}\, \exp\left(-\frac{\lambda\kappa^2}{2\alpha^2}\right). \tag{3.10}$$

In particular, the Gaussian $\mathcal{N}$ is in the kernel of the self-adjoint part of $Q$. We combine our two previous observations to define the function:

$$\mu(\theta, \kappa) = C \exp\left(-\frac{\lambda}{\alpha^2} H\right) = C \exp\left(-\frac{\lambda}{\alpha^2}\left(\frac{\kappa^2}{2} - \lambda\cos\theta\right)\right), \tag{3.11}$$

where $C$ is the normalization constant such that $\int_{(\theta,\kappa)} \mu(\theta, \kappa)\, d\theta\, d\kappa = 1$. This normalization constant is explicitly given below. The translates $\mu_{\overline{\theta}}$ of $\mu$ in the sense of definition (3.7) are of the form $g(H_{\overline{\theta}})$ and are Gaussian distributions in $\kappa$ with variance $\alpha/\sqrt{\lambda}$. It follows from a simple computation that $\mu_{\overline{\theta}}$ is an equilibrium for $Q$ (i.e. $Q(\mu_{\overline{\theta}}) = 0$), for all real values of $\overline{\theta}$.

To simplify the analysis, we introduce the Von Mises distribution $\mathcal{M}$:

$$\mathcal{M}(\theta) = C_0 \exp\left(\frac{\lambda^2}{\alpha^2}\cos\theta\right), \tag{3.12}$$

where $C_0 = (2\pi I_0(\frac{\lambda^2}{\alpha^2}))^{-1}$ is the normalization constant such that $\int_\theta \mathcal{M}(\theta)\, d\theta = 1$ (with $I_0$ the modified Bessel function of order 0). Therefore, $\mu$ can be written as the product of $\mathcal{M}$ given by (3.12) and $\mathcal{N}$ given by (3.10):

$$\mu(\theta,\kappa) = \mathcal{M}(\theta)\mathcal{N}(\kappa), \tag{3.13}$$

and the normalization constant $C$ is given by $C = C_0\sqrt{\lambda/(2\pi\alpha^2)}$. We summarize our analysis of $Q$ in the following proposition.

**Proposition 3.2** *i) The operator $Q$ satisfies:*

$$\int_{\theta,\kappa} Q(f)\,\frac{f}{\mu_{\overline{\theta}}}\, d\theta d\kappa = -\alpha^2 \int_{\theta,\kappa} \frac{\mathcal{N}}{\mathcal{M}_{\overline{\theta}}}\left|\partial_\kappa\left(\frac{f}{\mathcal{N}}\right)\right|^2 d\theta d\kappa \leq 0, \tag{3.14}$$

*with $\mu$ defined by (3.13) and $\overline{\theta}$ such that $\vec{\tau}(\overline{\theta}) = \Omega_f$ with $\Omega_f$ defined in (3.6).*

*ii) The equilibria of $Q$ (i.e. the functions $f(\theta,\kappa) \geq 0$ such that $Q(f) = 0$) form a two-dimensional manifold $\mathcal{E}$ given by:*

$$\mathcal{E} = \{\rho\,\mu_{\overline{\theta}} \quad | \quad \rho \in \mathbb{R}^+\,, \overline{\theta} \in (-\pi,\pi]\}, \tag{3.15}$$

*where $\rho$ is the total mass and $\overline{\theta}$ the direction of the flux of $\rho\,\mu_{\overline{\theta}}$.*

**Proof.** (i) Combining (3.8) and (3.9), we find:

$$Q(f) = \{H_{\overline{\theta}},\, f\} + \alpha^2\partial_\kappa\left(\mathcal{N}\partial_\kappa\left(\frac{f}{\mathcal{N}}\right)\right). \tag{3.16}$$

Using (3.11), the fact that the Poisson bracket with $f$ is a derivation and is a skew-adjoint operator, we find:

$$\begin{aligned}\int_{\theta,\kappa}\{H_{\overline{\theta}}, f\}\,\frac{f}{\mu}\, d\theta d\kappa &= \int_{\theta,\kappa}\frac{\alpha^2}{\lambda}\mathrm{e}^{-\frac{\lambda}{\alpha^2}H_{\overline{\theta}}}\{\mathrm{e}^{\frac{\lambda}{\alpha^2}H_{\overline{\theta}}}, f\}\frac{f}{\mu}\, d\theta d\kappa \\ &= \frac{\alpha^2}{\lambda}\frac{1}{C}\int_{\theta,\kappa}\{\mathrm{e}^{\frac{\lambda}{\alpha^2}H_{\overline{\theta}}}, f\} f\, d\theta d\kappa = 0.\end{aligned}$$

Then, using the formulation of $Q$ in (3.16), we easily deduce the equality (3.14) by applying Green's formula.

(ii) If $f$ is an equilibrium for $Q$ (i.e. $Q(f)=0$) using the equality (3.14) we have:

$$\int_{\theta,\kappa} \frac{\mathcal{N}(\kappa)}{\mathcal{M}_{\overline{\theta}}(\theta)} \left| \partial_\kappa \left( \frac{f}{\mathcal{N}} \right) \right|^2 d\theta d\kappa = 0,$$

which means that $f$ is proportional to $\mathcal{N}$ as a function of $\kappa$. Therefore, we can write:

$$f(\theta,\kappa) = \varphi(\theta)\mathcal{N}(\kappa).$$

Using again that $f$ is an equilibrium, we have:

$$-\kappa\varphi'(\theta) + \lambda \sin(\overline{\theta}-\theta)\frac{\lambda\kappa}{\alpha^2}\varphi(\theta) = 0, \quad \text{for all } \kappa.$$

Solving this differential equation leads to $\varphi(\theta) = C\mathcal{M}_{\overline{\theta}}(\theta)$ with $\mathcal{M}$ given by (3.12). This yields $f = K\mathcal{M}_{\overline{\theta}}\mathcal{N}$ with $K \geq 0$ a constant which proves that $f$ is of the form $f = \rho\mu_{\theta_0}$, with $\rho \geq 0$ and $\theta_0 \in (-\pi,\pi]$.

Reciprocally, we show that a function of the form $f = \rho\mu_{\theta_0}$ with $\rho \geq 0$ and $\theta_0 \in (-\pi,\pi]$ is an equilibrium. For this purpose, the only thing to show is that the associated $\Omega_f = \tau(\overline{\theta})$ is such that $\overline{\theta} = \theta_0$. We compute

$$\begin{aligned} \mathbf{j}_f &= \int_{(\theta,\kappa)} \rho\,\mu_{\theta_0}\,\tau(\theta)\,d\theta\,d\kappa \\ &= \rho \int_{(\theta,\kappa)} \mathcal{N}(\kappa)\,C_0\,\exp\left(\frac{\lambda^2}{\alpha^2}\cos(\theta-\theta_0)\right) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} d\theta\,d\kappa. \end{aligned}$$

Then, by the change of variables $\phi = \theta - \theta_0$ and using oddness considerations, we obtain

$$\mathbf{j}_f = \rho C_0 \int_\kappa \exp\left(\frac{\lambda^2}{\alpha^2}\cos\phi\right)\cos\phi\,d\phi\;\tau(\theta_0) = \rho\frac{I_1(\frac{\lambda^2}{\alpha^2})}{I_0(\frac{\lambda^2}{\alpha^2})}\,\tau(\theta_0),$$

where $I_1$ is the modified Bessel function of order 1. Remembering that $\Omega_f = \mathbf{j}_f/|\mathbf{j}_f|$, we deduce that $\overline{\theta} = \pm\theta_0$, with the sign being that of $I_1(\frac{\lambda^2}{\alpha^2})/I_0(\frac{\lambda^2}{\alpha^2})$. A simple inspection of the integral giving $I_1$ shows that this sign is positive and that $\overline{\theta} = \theta_0$, which ends the proof. □

### 3.3.2 Generalized collisional invariant

The next step to determine the hydrodynamic limit of $f^\varepsilon$ (3.4) is to look at the collision invariants of the operator $Q$, i.e. the functions $\psi$ which satisfy:

$$\int_{\theta,\kappa} Q(f)\psi\,d\theta d\kappa = 0, \qquad \text{for all } f.$$

Clearly, $\psi = 1$ is collisional invariant. But there is no other obvious collisional invariant. However, since the equilibria of $Q$ (3.15) form a two dimensional space, we need two conserved quantities to derive a macroscopic model. To overcome this problem, we use the notion of *generalized collisional invariant* developed in [20].

In this paper, we use a slightly different definition from [20]. Indeed, the result of [20] was slightly incorrect and the present definition is designed to make the statement correct. We first introduce the following definition:

**Definition 1** *For a given $\Omega \in \mathbb{S}^1$ and a given distribution function $f(\theta,\kappa)$, we define the 'extended' collision operator $\mathcal{Q}_\Omega(f)$ by:*

$$\mathcal{Q}_\Omega(f) = \{H_\Omega,\, f\} + \alpha^2 \partial_\kappa \left( \mathcal{N} \partial_\kappa \left( \frac{f}{\mathcal{N}} \right) \right),$$

*where we recall the notation (3.7).*

Obviously, we have

$$Q(f) = \mathcal{Q}_{\Omega_f}(f), \tag{3.17}$$

recalling the definition (3.6) of $\Omega_f$. For fixed $\Omega$, the operator $\mathcal{Q}_\Omega(f)$ is linear. We now define a Generalized Collision Invariant.

**Definition 2** *For a given unit vector $\Omega \in \mathbb{S}^1$, a function $\psi_\Omega$ is called a Generalized Collisional Invariant (GCI) if it satisfies:*

$$\int_{\theta,\kappa} \mathcal{Q}_\Omega(f)\, \psi_\Omega \, d\theta \, d\kappa = 0, \qquad \textit{for all } f \textit{ such that } \Omega_f = \pm\Omega, \tag{3.18}$$

Using definition (3.18) with $\Omega_f = \Omega$ and (3.17), we note that if $\psi_\Omega$ is a GCI, it satisfies

$$\int_{\theta,\kappa} Q(f)\, \psi_{\Omega_f} \, d\theta \, d\kappa = 0.$$

This property is crucial for the establishment of the hydrodynamic limit.

For a given $\Omega \in \mathbb{S}^1$, the adjoint operator to $\mathcal{Q}_\Omega$ is given by:

$$\mathcal{Q}^*_\Omega(\psi) = \kappa \partial_\theta \psi + \lambda \sin(\overline{\theta} - \theta)\, \partial_\kappa \psi - \lambda \kappa \partial_\kappa \psi + \alpha^2 \partial^2_\kappa \psi,$$

with $\overline{\theta}$ such that $\Omega = \vec{\tau}(\overline{\theta})$. This operator $\mathcal{Q}^*_\Omega$ enables us to find an explicit equation for the GCI $\psi_\Omega$ as stated in the following lemma.

**Lemma 3.3** *For a given unit vector $\Omega \in \mathbb{S}^1$, a function $\psi_\Omega$ is a generalized collisional invariant if and only if it there exists a constant $\beta \in \mathbb{R}$ such that:*

$$\mathcal{Q}^*_\Omega(\psi_\Omega) = \beta\, \vec{\tau}(\theta) \times \Omega. \tag{3.19}$$

**Proof.** Let $f(\theta,\kappa)$ be such that $\Omega_f = \pm\Omega$. This is equivalent to saying that there exists a constant $C \in \mathbb{R}$ such that $\mathbf{j}_f = C\Omega$ (see (3.6) for the definition of $\mathbf{j}_f$), or in other words, that $\mathbf{j}_f \times \Omega = 0$. Now, if $\psi$ satisfies (3.19), we have, for such a function $f$:

$$\begin{aligned}\int_{\theta,\kappa} \mathcal{Q}_\Omega(f)\,\psi\,d\theta d\kappa &= \int_{\theta,\kappa} f\,Q^*_\Omega(\psi)\,d\theta d\kappa \\ &= \beta\int_{\theta,\kappa} f\vec{\tau}(\theta)\times\Omega\,d\theta d\kappa = \beta\,\mathbf{j}_f\times\Omega = 0,\end{aligned}$$

and $\psi$ is a GCI associated to $\Omega$.
Reciprocally, if $\psi_\Omega$ is a GCI associated to $\Omega$, we have:

$$\int_{\theta,\kappa} \mathcal{Q}_\Omega(f)\,\psi_\Omega\,d\theta\,d\kappa = 0 = \int_{\theta,\kappa} f\,\mathcal{Q}^*_\Omega(\psi_\Omega)\,d\theta d\kappa$$

for all $f(\theta,\kappa)$ such that $\mathbf{j}_f \times \Omega = 0$. We deduce that, for all $f$,

$$\mathbf{j}_f \times \Omega = 0 \quad \Longrightarrow \quad \int_{\theta,\kappa} f Q^*_\Omega(\psi_\Omega)\,d\theta d\kappa = 0. \tag{3.20}$$

The two expressions appearing in (3.20) are linear forms acting on $f$. By an elementary lemma [6], the one appearing in the right-hand side is proportional to the one appearing in the left-hand side, with a proportionality coefficient $\beta \in \mathbb{R}$. Expressing this proportionality gives:

$$\int_{\theta,\kappa} f(Q^*_\Omega(\psi_\Omega) - \vec{\tau}(\theta)\times\Omega)\,d\theta d\kappa = 0, \tag{3.21}$$

for all $f$ without any restriction. (3.21) yields (3.19), which concludes the proof. □

It remains to prove the existence of GCI's, or, in other words, to prove the existence of solutions to equation (3.19). With this aim, we use the Hilbert space $L^2_\mu$ equipped with the scalar product $\langle .,. \rangle_\mu$ defined by:

$$\begin{aligned}&L^2_\mu = \{f(\theta,\kappa) \;/\; \int_{\theta,\kappa} |f|^2\,\mu\,d\theta d\kappa < +\infty\}, \\ &\langle f,\, g\rangle_\mu = \int_{\theta,\kappa} fg\,\mu\,d\theta d\kappa.\end{aligned} \tag{3.22}$$

Below, we will also use the notation:

$$\langle g\rangle_\mu = \int_{\theta,\kappa} g(\theta,\kappa)\,\mu(\theta,\kappa)\,d\theta d\kappa. \tag{3.23}$$

We define the hyperplane $E$:

$$E = \{f \in L^2_\mu(\theta,\kappa) \;/\; \int_{\theta,\kappa} f\,\mu\,d\theta d\kappa = 0\}$$

and the linear operator $\mathcal{L}$:

$$\mathcal{L}\psi = \kappa\partial_\theta\psi - \lambda\sin\theta\partial_\kappa\psi - \lambda\kappa\partial_\kappa\psi + \alpha^2\partial_\kappa^2\psi \tag{3.24}$$

with domain $D(\mathcal{L})$ given by:

$$D(\mathcal{L}) = \{f \in L^2_\mu \ / \ \mathcal{L}f \in L^2_\mu\}.$$

We have the following lemma:

**Lemma 3.4** *(i) Let $\chi \in L^2_\mu$. A necessary condition for the existence of a solution $\psi \in D(\mathcal{L})$ of problem*

$$\mathcal{L}\psi = \chi, \tag{3.25}$$

*is that $\chi \in E$ or in other words, that $\chi$ satisfies the solvability condition $\int_{\theta,\kappa}\chi\,\mu\,d\theta d\kappa = 0$.*
*(ii) For all $\chi \in E$, the problem (3.25) has a unique solution $\psi$ in $E$. Then, all solutions to problem (3.25) are of the form $\psi + K$, with an arbitrary $K \in \mathbb{R}$.*

In Appendices A1 and A2, we give two different proofs of the fact that (3.25) is uniquely solvable in $E$. The proof in appendix A1 uses tools from functional analysis (see also [21]). The proof in appendix A2 uses probabilistic tools to analyze the stochastic equation associated to (3.25) (see also [10]). Here we only prove (i) and the last statement of (ii).

**Proof.** (i) The formal adjoint $\mathcal{L}^*$ of $\mathcal{L}$ is given by the expression (3.5) of $Q$ in which $\overline{\theta} = 0$. Therefore, from section 3.3.1, we have that $\mathcal{L}^*(\mu) = 0$. Integrating (3.25) against $\mu$ and using Green's formula leads to the necessary condition $\int_{\theta,\kappa}\chi\,\mu\,d\theta d\kappa = 0$, i.e. to the fact that $\chi$ must belong to $E$.

The second part of (ii) amounts to showing that the null space of $\mathcal{L}$ reduces to the constant functions. Indeed, it is straightforward to see that $\mathcal{L}(1) = 0$. To prove that the constant functions are the only elements of the null space of $\mathcal{L}$, we suppose that $\psi \in D(\mathcal{L})$ such that $\mathcal{L}\psi = 0$. Using that $\langle\mathcal{L}\psi,\,\psi\rangle_\mu = 0$, we find, using Green's formula:

$$\int_{\theta,\kappa} |\partial_\kappa\psi|^2\,\mu\,d\theta d\kappa = 0.$$

Therefore, $\psi$ is independent of $\kappa$. So we can write: $\psi(\theta,\kappa) = \Phi(\theta)$. Using again that $\mathcal{L}\Phi = 0$, we find that $\Phi$ is a constant.

We refer to appendices A1 or A2 for the existence part of point (ii). □

The following proposition completely determines the set of GCI's associated to a vector $\Omega$.

**Proposition 3.5** *For a given $\Omega \in \mathbb{S}^1$, the set $C_\Omega$ of the GCI's associated to $\Omega$ is a two dimensional vector space $C_\Omega = Span\{1, \psi_\Omega\}$ where $\psi_\Omega$ is given by:*

$$\psi_\Omega(\theta, \kappa) = \psi(\theta - \overline{\theta}, \kappa), \tag{3.26}$$

*with $\overline{\theta}$ such that $\vec{\tau}(\theta) = \Omega$ and $\psi$ is the unique solution of:*

$$\mathcal{L}\psi = -\sin\theta, \tag{3.27}$$

*belonging to the hyperplane $E$. Moreover, the function $\psi$ satisfies the property:*

$$\psi(-\theta, -\kappa) = -\psi(\theta, \kappa). \tag{3.28}$$

**Proof.** We first note that (3.19) is a linear problem and that it is enough to solve it for $\beta = 1$. Simple calculations show that $\psi_\Omega$ is a solution to (3.19) if and only if there exists a function $\psi$ such that $\psi_\Omega(\theta) = \psi(\theta - \overline{\theta})$ with $\psi$ a solution of (3.27). This shows (3.26).

To show the existence and uniqueness of a solution $\psi$ to (3.27) in $E$, it is enough to check that the right-hand side of (3.27) belongs to $E$ i.e. satisfies the compatibility condition $\int_{\theta,\kappa} \chi\, \mu\, d\theta d\kappa = 0$. But this follows readily by oddness considerations. Moreover, noting that the operator $\mathcal{L}$ is invariant under the transformation $(\theta, \kappa) \to (-\theta, -\kappa)$, (3.28) follows from the uniqueness of the solution.

Again, by the uniqueness in $E$ and by the second part of Lemma 3.4 (ii), all solutions to (3.27) consist of linear combinations of $\psi$ and of a constant function. It follows that the set of GCI's associated to $\Omega$ is the two-dimensional vector space spanned $C_\Omega = Span\{1, \psi_\Omega\}$. This ends the proof. □

## 3.4 Limit $\varepsilon \to 0$

Since we know the equilibria and GCI's of the operator $Q$, we can give a formal proof of theorem 1.

**Proof of Theorem 1.** If we suppose that $f^\varepsilon$ converges (weakly) to $f^0$ as $\varepsilon \to 0$ we first have:

$$Q(f^0) = 0,$$

which means that $f^0$ is an equilibrium. Thanks to section 3.3.1, $f^0$ can be written as:

$$f^0 = \rho^0 \mathcal{M}_{\Omega^0}(\theta)\mathcal{N}(\kappa),$$

with $\mathcal{M}$ and $\mathcal{N}$ defined in (3.12) (3.10). The mass $\rho^0(t, \mathbf{x})$ and the direction of the flux $\Omega^0(t, \mathbf{x})$ are the two remaining unknowns.

In order to find the system of equations which determines the evolution of $\rho^0$ and $\Omega^0$, we first integrate (3.4) with respect to $(\theta, \kappa)$. We find the mass conservation equation:

$$\partial_t \rho^\varepsilon + \nabla_{\mathbf{x}} \cdot \mathbf{j}^\varepsilon = 0,$$

with

$$\mathbf{j}^\varepsilon = \int_{\theta,\kappa} \vec{\tau}(\theta) f^\varepsilon \, d\theta d\kappa.$$

In the limit $\varepsilon \to 0$, this gives:

$$\mathbf{j}^\varepsilon \stackrel{\varepsilon \to 0}{\longrightarrow} \mathbf{j}^0 = c_1 \rho^0 \Omega^0,$$

with the constant $c_1$ given by:

$$c_1 = \int_\theta \cos\theta \, \mathcal{M}(\theta) \, d\theta = \frac{I_1(\frac{\lambda^2}{\alpha^2})}{I_0(\frac{\lambda^2}{\alpha^2})}. \tag{3.29}$$

Therefore we deduce that $\rho^0$ and $\Omega^0$ obey the following mass conservation equation:

$$\partial_t \rho^0 + c_1 \nabla_{\mathbf{x}} \cdot (\rho^0 \Omega^0) = 0.$$

In order to fully determine the evolution of $\rho^0$ and $\Omega^0$, we need to find a second equation. For this purpose, we integrate (3.4) against the generalized collisional invariant $\psi_{\Omega^\varepsilon}$ (3.26), with $\Omega^\varepsilon = \Omega_{f^\varepsilon}$. This leads to:

$$\int_{\theta,\kappa} (\partial_t f^\varepsilon + \vec{\tau}(\theta) \cdot \nabla_x f^\varepsilon) \psi_{\Omega^\varepsilon} \, d\theta d\kappa = 0.$$

In the limit $\varepsilon \to 0$, we find :

$$\int_{\theta,\kappa} \partial_t (\rho^0 \mathcal{M}_{\Omega^0} \mathcal{N}) \, \psi_{\Omega^0} \, d\theta d\kappa + \int_{\theta,\kappa} \vec{\tau}(\theta) \cdot \nabla_{\mathbf{x}} (\rho^0 \mathcal{M}_{\Omega^0} \mathcal{N}) \, \psi_{\Omega^0} \, d\theta d\kappa = 0. \tag{3.30}$$

For clarity, we drop the exponent '0' and write $(\rho, \Omega)$ for $(\rho^0, \Omega^0)$ in the discussion below. Using polar coordinates for $\Omega = \vec{\tau}(\overline{\theta}) = (\cos\theta, \sin\theta)$, elementary computations show that:

$$\begin{aligned} \partial_t(\rho \mathcal{M}_{\overline{\theta}}) + \vec{\tau}(\theta) \cdot \nabla_{\mathbf{x}} (\rho \mathcal{M}_{\overline{\theta}}) &= \partial_t \rho \, \mathcal{M}_{\overline{\theta}} + \rho \mathcal{M}_{\overline{\theta}} \frac{\lambda^2}{\alpha^2} \sin(\theta - \overline{\theta}) \partial_t \overline{\theta} \\ &\quad + \vec{\tau}(\theta) \cdot \Big( \nabla_{\mathbf{x}} \rho \mathcal{M}_{\overline{\theta}} + \rho \mathcal{M}_{\overline{\theta}} \frac{\lambda^2}{\alpha^2} \sin(\theta - \overline{\theta}) \nabla_{\mathbf{x}} \overline{\theta} \Big). \end{aligned}$$

Therefore, equation (3.30) leads to:

$$\begin{aligned}
& \int_{\theta,\kappa} \partial_t \rho \, \mathcal{M}_{\overline{\theta}} \mathcal{N} \, \psi_{\overline{\theta}} \, d\theta d\kappa \\
+ \quad & \frac{\lambda^2}{\alpha^2} \int_{\theta,\kappa} \rho \, \mathcal{M}_{\overline{\theta}} \mathcal{N} \, \sin(\theta - \overline{\theta}) \, \partial_t \overline{\theta} \, \psi_{\overline{\theta}} \, d\theta d\kappa \\
+ \quad & \int_{\theta,\kappa} \vec{\tau}(\theta) \cdot \left( \nabla_{\mathbf{x}} \rho \, \mathcal{M}_{\overline{\theta}} \mathcal{N} \, \psi_{\overline{\theta}} \right) d\theta d\kappa \\
+ \quad & \frac{\lambda^2}{\alpha^2} \int_{\theta,\kappa} \vec{\tau}(\theta) \cdot \left( \rho \, \mathcal{M}_{\overline{\theta}} \mathcal{N} \, \sin(\theta - \overline{\theta}) \, \nabla_{\mathbf{x}} \overline{\theta} \, \psi_{\overline{\theta}} \right) d\theta d\kappa = 0.
\end{aligned}$$

This equation can be simplified using the symmetry satisfied by $\psi$ (3.28). We treat each term separately. First, we have:

$$\begin{aligned}
X_1 &= \int_{\theta,\kappa} \partial_t \rho \, \mathcal{M}_{\overline{\theta}} \mathcal{N} \, \psi_{\overline{\theta}} \, d\theta d\kappa \\
&= \partial_t \rho \int_{\theta,\kappa} \mathcal{M}(\theta - \overline{\theta}) \mathcal{N}(\kappa) \, \psi(\theta - \overline{\theta}, \kappa) \, d\theta d\kappa = 0, \qquad (3.31)
\end{aligned}$$

because $\mathcal{M}(\theta)\mathcal{N}(\kappa)$ is an even function of the pair $(\theta, \kappa)$ and $\psi(\theta, \kappa)$ is odd. For the second term, we use the change of unknowns $\theta' = \theta - \overline{\theta}$ and get:

$$\begin{aligned}
X_2 &= \frac{\lambda^2}{\alpha^2} \rho \, \partial_t \overline{\theta} \int_{\theta',\kappa} \mathcal{M}(\theta') \mathcal{N}(\kappa) \sin\theta' \, \psi(\theta', \kappa) \, d\theta' d\kappa \\
&= \frac{\lambda^2}{\alpha^2} \rho \, \partial_t \overline{\theta} \, \gamma_1, \qquad (3.32)
\end{aligned}$$

with

$$\gamma_1 = \langle \sin\theta \, \psi \rangle_\mu \qquad (3.33)$$

using the notation (3.23). For the third term, we find:

$$\begin{aligned}
X_3 &= \nabla_{\mathbf{x}} \rho \cdot \int_{\theta,\kappa} \vec{\tau}(\theta) \, \mathcal{M}_{\overline{\theta}} \mathcal{N} \, \psi_{\overline{\theta}} \, d\theta d\kappa \\
&= \nabla_{\mathbf{x}} \rho \cdot \int_{\theta,\kappa} \vec{\tau}(\theta + \overline{\theta}) \, \mathcal{M}(\theta) \mathcal{N}(\kappa) \, \psi(\theta, \kappa) \, d\theta d\kappa \\
&= \nabla_{\mathbf{x}} \rho \cdot \int_{\theta,\kappa} \begin{pmatrix} \cos\theta \cos\overline{\theta} - \sin\theta \sin\overline{\theta} \\ \sin\theta \cos\overline{\theta} + \cos\theta \sin\overline{\theta} \end{pmatrix} \mathcal{M}(\theta) \mathcal{N}(\kappa) \, \psi(\theta, \kappa) \, d\theta d\kappa.
\end{aligned}$$

Once again, using the symmetry satisfied by $\psi$, we find:

$$X_3 = \gamma_1 \nabla_{\mathbf{x}} \rho \cdot \begin{pmatrix} -\sin\overline{\theta} \\ \cos\overline{\theta} \end{pmatrix},$$

with $\gamma_1$ defined in (3.33). If we denote by $\vec{\tau}(\overline{\theta})^\perp = \Omega^\perp$ the orthogonal vector to $\vec{\tau}(\overline{\theta})$:

$$\vec{\tau}(\overline{\theta})^\perp = \Omega^\perp = \begin{pmatrix} -\sin\overline{\theta} \\ \cos\overline{\theta} \end{pmatrix},$$

we finally get:

$$X_3 = \gamma_1 \nabla_{\mathbf{x}}\rho \cdot \vec{\tau}(\overline{\theta})^\perp. \tag{3.34}$$

For the last term, we have:

$$\begin{aligned} X_4 &= \frac{\lambda^2}{\alpha^2}\,\rho\,\nabla_{\mathbf{x}}\overline{\theta}\cdot\int_{\theta,\kappa}\vec{\tau}(\theta)\mathcal{M}_{\overline{\theta}}(\theta)\mathcal{N}(\kappa)\sin(\theta-\overline{\theta})\psi_{\overline{\theta}}(\theta,\kappa)\,d\theta d\kappa \\ &= \frac{\lambda^2}{\alpha^2}\,\rho\,\nabla_{\mathbf{x}}\overline{\theta}\cdot\int_{\theta,\kappa}\vec{\tau}(\theta+\overline{\theta})\mathcal{M}_0(\theta)\mathcal{N}(\kappa)\sin\theta\,\psi(\theta,\kappa)\,d\theta d\kappa \\ &= \frac{\lambda^2}{\alpha^2}\,\gamma_2\,\rho\,\nabla_{\mathbf{x}}\overline{\theta}\cdot\vec{\tau}(\theta), \end{aligned} \tag{3.35}$$

with

$$\gamma_2 = \langle\cos\theta\,\sin\theta\,\psi\rangle_\mu.$$

Combining (3.31), (3.32), (3.34) and (3.35) yields:

$$\gamma_1\,\frac{\lambda^2}{\alpha^2}\,\rho\,\partial_t\overline{\theta} + \gamma_1\nabla_{\mathbf{x}}\rho\cdot\vec{\tau}(\overline{\theta})^\perp \;+\; \gamma_2\frac{\lambda^2}{\alpha^2}\,\rho\,\nabla_{\mathbf{x}}\overline{\theta}\cdot\vec{\tau}(\theta) = 0. \tag{3.36}$$

Using again the unit vector $\Omega = \vec{\tau}(\overline{\theta})$, elementary computations show that:

$$\partial_t\Omega = \partial_t\overline{\theta}\,\Omega^\perp \qquad \text{and} \qquad (\Omega\cdot\nabla_{\mathbf{x}})\Omega = (\Omega^\perp\otimes\Omega)\nabla_{\mathbf{x}}\overline{\theta}.$$

Therefore, multiplying equation (3.36) by $\Omega^\perp$ leads to:

$$\rho\,\partial_t\Omega \;+\; \frac{\alpha^2}{\lambda^2}\,(\nabla_{\mathbf{x}}\rho\cdot\Omega^\perp)\Omega^\perp \;+\; \frac{\gamma_2}{\gamma_1}\,\rho\,(\Omega\cdot\nabla_{\mathbf{x}})\Omega = 0.$$

This finally leads to:

$$\rho\,\partial_t\Omega \;+\; c_2\;\rho\,(\Omega\cdot\nabla_{\mathbf{x}})\Omega \;+\; \frac{\alpha^2}{\lambda^2}\,(\mathrm{Id}-\Omega\otimes\Omega)\nabla_{\mathbf{x}}\rho = 0, \tag{3.37}$$

with

$$c_2 = \frac{\gamma_2}{\gamma_1} = \frac{\langle\sin\theta\cos\theta\,\psi\rangle_\mu}{\langle\sin\theta\,\psi\rangle_\mu}, \tag{3.38}$$

which ends the proof. □

# 4 Properties of the macroscopic system

## 4.1 Hyperbolicity

The macroscopic system (2.13) arising from the PTWA dynamics has the same form as the system found in [20] for the macroscopic limit of the Vicsek model. Indeed, if we define the diffusion coefficient $d$ as:

$$d = \frac{\alpha^2}{\lambda^2},$$

then the coefficient $c_1$ given by (3.29) and the coefficient $\frac{\alpha^2}{\lambda^2}$ in front of the pressure term in (3.37) are exactly the same in the two systems. Only the coefficient $c_2$ given by (3.38) differs from that of [20]. Thus, the study of the hyperbolicity of system (2.13) is completely similar to the one conducted for the Vicsek model in [20, 33]. We briefly summarize the analysis here. Using the geometric constraint $|\Omega| = 1$, we can parametrize the direction of the flux $\Omega$ in polar coordinates: $\Omega = (\cos\theta, \sin\theta)$ with $\theta \in ]-\pi, \pi]$. In order to look at the wave propagating in the $x$-direction, we suppose that $\rho$ and $\Omega$ are independent of $y$. Therefore, under this assumption, the system (2.13) reduces to:

$$\begin{aligned} \partial_t \rho + c_1 \partial_x (\rho \cos\theta) &= 0, \\ \partial_t \theta + c_2 \cos\theta \partial_x \theta - \frac{\alpha^2}{\lambda^2} \frac{\sin\theta}{\rho} \partial_x \rho &= 0. \end{aligned}$$

The characteristic velocities of this system are given by:

$$\gamma = \frac{1}{2} \left[ (c_1 + c_2)\cos\theta \pm \sqrt{(c_1 - c_2)^2 \cos^2\theta + 4c_1 \frac{\alpha^2}{\lambda^2} \sin^2\theta} \right].$$

The system is therefore *hyperbolic* since the characteristic velocities are real.

## 4.2 Numerical computations of $\psi$

In order to compute the macroscopic coefficient $c_2$ (3.38), we first need to calculate the generalized collisional invariant $\psi$ (3.26). With this aim, we introduce a weak formulation of the equation satisfied by $\psi$. In the Hilbert space $L^2_\mu(\mathbb{S}^1 \times \mathbb{R})$, the function $\psi$ satisfies:

$$\langle \mathcal{L}\psi, \varphi \rangle_\mu = -\langle \sin\theta, \varphi \rangle_\mu , \quad \forall \varphi \in L^2_\mu, \tag{4.1}$$

where the scalar product $\langle ., . \rangle_\mu$ is defined in (3.22) and the operator $\mathcal{L}$ in (3.24). To approximate the solution $\psi$ numerically, we use a Galerkin method. It consists in solving the weak formulation (4.1) for all the functions $\varphi$ in a subspace $V$ of $L^2_\mu$ of finite dimension. To construct such a subspace $V$, we use a Hilbert basis of $L^2_\mu$. For this purpose, we consider the following functions:

$$\varphi_m(\theta) = \frac{e^{im\theta}}{\sqrt{2\pi\mathcal{M}(\theta)}} \quad , \quad P_n(\kappa) = \frac{H_n\left(\frac{\sqrt{\lambda}}{\alpha}\kappa\right)}{\sqrt{n!}},$$

where $\mathcal{M}$ is defined in (3.12) and $H_n$ is the n$^{\text{th}}$ Hermite polynomial. We can easily prove that the family $\{\varphi_m P_n\}_{m,\,n\geq 0}$ is a Hilbert basis of $L^2_\mu$. Then, for any odd positive integers $m$ and any positive integer $n$, we define the vector space $V_{m,n}$:

$$V_{m,n} = \mathrm{Span}\{\varphi_j P_k \;/\; |j| \leq m \;,\; 0 \leq \; k \; \leq n\}.$$

The Galerkin method consists in finding $\psi_{m,n} \in V_{m,n}$ such that equation (4.1) is satisfied for every $\varphi \in V_{m,n}$:

$$\langle \mathcal{L}\psi_{m,n},\, \varphi\rangle_\mu = -\langle \sin\theta,\, \varphi\rangle_\mu \;, \quad \forall\, \varphi \in V_{m,n}. \tag{4.2}$$

We can decompose $\psi$ as:

$$\psi_{m,n}(\theta,\kappa) = \sum_{|j|<m, 0\leq k\leq n} C_j^k \varphi_m(\theta) P_n(\kappa), \tag{4.3}$$

where $C_j^k$ are complex coefficients given by:

$$C_j^k = \langle \psi_{m,n},\, \varphi_j\, P_k\rangle_\mu.$$

We store the coefficients $\{C_j^k\}_{|j|\leq m, 0\leq k\leq n}$ in a matrix $X$ such that:

$$X(j,k) = C_j^k. \tag{4.4}$$

We call the matrix $X$ the matrix representation of $\psi_{m,n}$ in $V_{m,n}$. We want to transform the problem satisfied by $\psi_{m,n}$ (4.1) into a matrix equation for $X$. With this aim, we define several matrices.

**Definition 3** *We define the matrices $L_{-1}$ and $L_{+1}$ by:*

$$L_{-1} = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad L_{+1} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \tag{4.5}$$

*and the diagonal matrices:*

$$\begin{aligned} D_1 &= diag(-m,\, \dots,\, -1,\, 0,\, 1,\, \dots,\, m) \\ D_2 &= diag(0,\, 1,\, 2,\, \dots\, n). \end{aligned}$$

Using the matrices defined above, we can convert the equation satisfied by $\psi_{m,n}$ (4.2) into a matrix equation for $X$.

**Proposition 4.1** *Let $\psi_{m,n} \in V_{m,n}$ the solution of (4.1) in $V_{m,n}$. Its matrix representation $X = \{C_j^k\}_{|j|\le m, 0\le k\le n}$ in the Hilbert basis $\{\varphi_m P_n\}$ satisfies:*

$$\beta_1 M_1 X N_1 + \beta_2 M_2 X N_2 - \lambda X D_2 = B \tag{4.6}$$

*with*

$$\begin{array}{rclcrcl} \beta_1 &=& \frac{i\alpha}{\sqrt{\lambda}} &,& \beta_2 &=& \frac{i\lambda\sqrt{\lambda}}{4\alpha}, \\ M_1 &=& D_1 &,& N_1 &=& \sqrt{D_2}L_{-1} + L_{+1}\sqrt{D_2}, \\ M_2 &=& L_{-1} - L_{+1} &,& N_2 &=& \sqrt{D_2}L_{-1} - L_{+1}\sqrt{D_2}. \end{array} \tag{4.7}$$

*and $B$ the matrix representation of $-\sin\theta$ in $V_{m,n}$ given by:*

$$B(j,k) = \begin{cases} \frac{i}{2\sqrt{I_o(\frac{\lambda^2}{\alpha^2})}} \left( I_{|j-1|}\left(\frac{\lambda^2}{2\alpha^2}\right) - I_{|j+1|}\left(\frac{\lambda^2}{2\alpha^2}\right)\right) & \text{if } k = 0, \\ 0 & \text{otherwise,} \end{cases}$$

*where $I_j$ is the modified Bessel function of order $j$.*

Since the demonstration of proposition 4.1 is only a matter of computations, we postpone the proof to appendix B. To solve (4.6), we transform the linear equation (4.6) into a linear system that we invert numerically. This eventually allows us to construct $\psi_{m,n}$ using (4.3).

On figure 2 (left), we display an example of an approximate solution $\psi_{m,n}$ of the GCI $\psi$ for $\lambda = 1$ and $\alpha = 1$. We also estimate $\mathcal{L}\psi_{m,n}$ numerically using a finite difference method (figure 2, right). The figure clearly suggests that $\mathcal{L}\psi_{m,n}$ is close to $-\sin\theta$, providing a qualitative check of the accuracy of the computation. To make this assessment more quantitative, we compute the residual $|\mathcal{L}\psi_{m,n} + \sin\theta|_\infty$ for different values of $(\lambda, \alpha)$ on figure 3. As we can see, the residual gets larger when $\alpha$ increases and gets smaller when $\lambda$ increases.

## 4.3 Computation of the coefficient $c_2$

Once we have computed the generalized collisional invariant $\psi$, we can calculate the coefficient $c_2$ using (3.38). On figure 4, we fix the parameter $\lambda = 1$ and we compute the value of $c_2$ for different values of $\alpha$ (we still use $m = 30$ and $n = 61$ to get a numerical approximation of $\psi_{m,n}$). In the same graph, we add show the coefficient $c_2$ of the Vicsek model [20,33] for $d = \frac{\alpha^2}{\lambda^2}$. The relative error between the two curves is very small (around 5%). This similarity between the two curves shows a strong connexion between the PTWA model and the Vicsek model. Work is in progress to study the link between the two models more deeply.

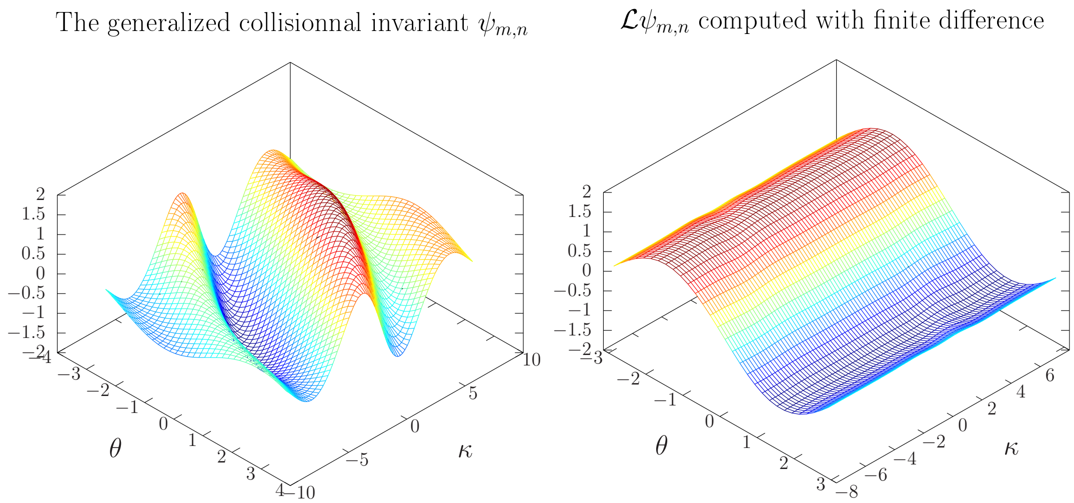


Figure 2: Left figure: the generalized collisional invariant $\psi_{m,n}$ for $\lambda = 1$ and $\alpha = 1$ computed using $m = 30$ and $n = 61$. Right figure: we compute $\mathcal{L}\psi_{m,n}$ using a finite difference method with $\Delta\theta = .2$ and $\Delta\kappa = .2$. We clearly recover the function $-\sin\theta$ (see figure 3 for a more detailed comparison).

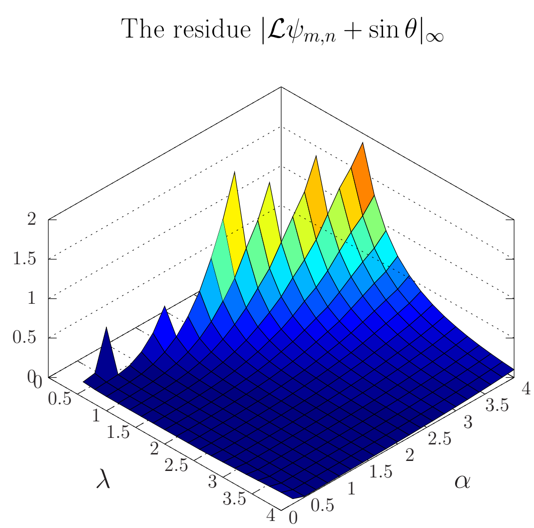


Figure 3: The residual $|\mathcal{L}\psi_{m,n} + \sin\theta|_\infty$ estimated on the interval $(\theta, \kappa) \in [-\pi, \pi] \times [-5, 5]$ for different values of $(\lambda, \alpha)$. $\psi_{m,n}$ is computed as in figure 2 (left) and $\mathcal{L}\psi_{m,n}$ is computed using a finite difference scheme (with $\Delta\theta = \Delta\kappa = .2$). The residual increases with $\alpha$ and decreases with $\lambda$.

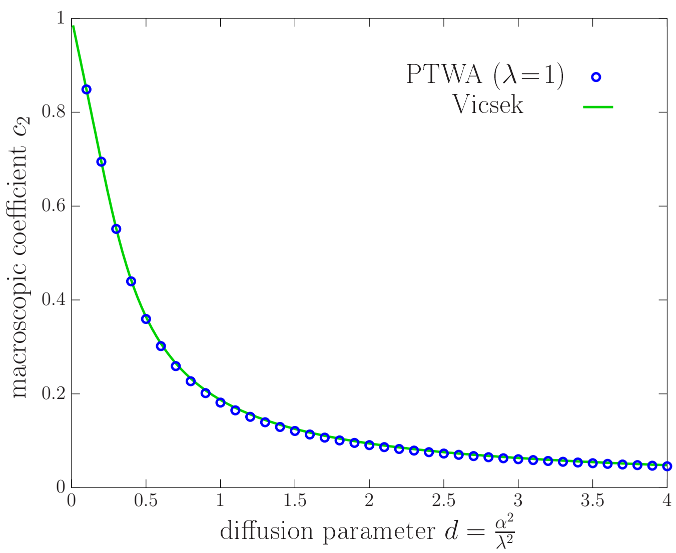


Figure 4: The coefficient $c_2$ in the PTWA model (3.38) computed for $\lambda = 1$ and different values of $\alpha$ (blue) and the coefficient $c_2$ in the Vicsek model (green). The relative error between the two curves is around 5%.

# 5 Conclusion

In this work, we have introduced a new Individual-Based Model describing the displacement of individuals which tend to align with theirs neighbors. This model, called 'Persistent Turning Walker model with Alignment' (PTWA), is a combination of the phenomenological Vicsek alignment model [41] with the experimentally derived PTW model of fish displacement [24]. We have established the macroscopic limit of this model within a hydrodynamic scaling where the radius of interaction of the agents is tied to the microscopic scale. The derivation uses a new notion of 'Generalized Collisional Invariant' developed earlier in [20]. The numerical computations of the coefficients involved in this macroscopic model have shown that there are important similarities between the PTWA model and the Vicsek model at large scale.

The present work proves that the addition of a local alignment rule in the PTW model changes drastically the large-scale dynamics as compared to the PTW model without alignment interaction. Indeed, while the PTW model without alignment is diffusive at large scales, the PTWA model becomes hyperbolic, of hydrodynamic type. As a summary, local alignment generates macroscopic convection.

In future work, the relation between the PTWA and Vicsek dynamics will be further explored, both at the microscopic and macroscopic levels. This ensemble of models forms a complex hierarchy. Numerical simulations and comparisons over a wide range of parameters will be performed to better understand the relations between these models.

Many questions concerning the derivation of macroscopic models remain open in this context. One possible route is to explore what the macroscopic limit of the PTWA model becomes when an attraction-repulsion rule is added. More generally, it may be possible to classify the different types of Individual-Based Models by looking at their corresponding macroscopic limits. Another direction is to quantify how close the macroscopic model is to the corresponding microscopic model. In particular, the question of determining what minimal number of individuals is required for the macroscopic description to be valid is of crucial importance. All these questions call for deeper numerical studies which will permit to understand when the microscopic and macroscopic descriptions are similar and when they are not.

# Appendix A1: Proof of lemma 3.4 (ii) (functional analytic proof)

**Proof.** First, we prove the uniqueness of the solution of (3.25) in $E$. Indeed, we have shown in section 3.3.2 that the null space $\ker(\mathcal{L})$ of $\mathcal{L}$ consists of the constant functions. Therefore, $\ker(\mathcal{L}) \cap E = \{0\}$, which shows the uniqueness of the solutions of (3.25) in $E$.

To prove the existence of a solution of (3.25), we first consider a slightly modified version of equation (3.25): for a given $\varepsilon > 0$, we want to solve

$$-\varepsilon\psi + \mathcal{L}\psi \;=\; \chi. \tag{A.1}$$

Thanks to this modification, we have the inequality:

$$\langle \varepsilon\psi - \mathcal{L}\psi,\, \psi\rangle_\mu = \varepsilon|\psi|_\mu^2 + \alpha^2|\partial_\kappa\psi|_\mu^2 \geq \varepsilon|\psi|_\mu^2.$$

Therefore the operator $\varepsilon Id - \mathcal{L}$ is coercive, so we can apply the theorem of J. L. Lions in [31] which gives a weak solution $\psi_\varepsilon$ in $E$ of the problem (A.1).

To find a solution of $\mathcal{L}\psi = \chi$, we need to extract a convergent subsequence of $\{\psi_\varepsilon\}_{\varepsilon>0}$ when $\varepsilon$ goes to zero. The limit will satisfy (3.25). Since $E$ is an Hilbert space, it remains to prove that the family $\{\psi_\varepsilon\}_{\varepsilon>0}$ is bounded in $E$. For that, we proceed by contradiction. If the family $\{\psi_\varepsilon\}_\varepsilon$ is not bounded in $E$ as $\varepsilon$ tends to 0, there exists a subsequence $\varepsilon_n$ such that:

$$|\psi_{\varepsilon_n}|_\mu \overset{n\to\infty}{\longrightarrow} +\infty \qquad , \qquad \varepsilon_n \overset{n\to\infty}{\longrightarrow} 0.$$

To simplify the notations, we use the subscript $\varepsilon$ for $\varepsilon_n$ in the following. Defining the functions:

$$U_\varepsilon = \frac{\psi_\varepsilon}{N_\varepsilon} \tag{A.2}$$

with $N_\varepsilon = |\psi_\varepsilon|_\mu$, we have that:

$$-\varepsilon U_\varepsilon + \mathcal{L}U_\varepsilon = \frac{\chi}{N_\varepsilon}.$$

Since the sequence $\{U_\varepsilon\}_\varepsilon$ is bounded ($|U_\varepsilon|_\mu = 1$), we can extract a weakly convergent subsequence (denoted by $\varepsilon$ once again) such that:

$$U_\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} U_0 \quad \text{weakly in } L^2_\mu.$$

In particular, since $N_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} +\infty$, we have that $\mathcal{L}U_0 = 0$ and therefore by uniqueness $U_0 = 0$. This means that $U_\varepsilon$ converges weakly to zero. We will obtain a contradiction with the fact $|U_\varepsilon|_\mu = 1$ if we prove that $U_\varepsilon$ converges strongly to zero.

To prove the strong convergence of $U_\varepsilon$, we decompose the functions $U_\varepsilon$ in two parts. For that, we introduce the vector space $L$:

$$L = \{\Phi \in L^2(\mathbb{S}^1) \;/\; \int_\theta \Phi(\theta)\, \mathcal{M}(\theta)\, d\theta = 0\}.$$

It is easy to see that $L \subset E$. We denote by $L^\perp$ the orthogonal space of $L$ such that:

$$E = L \overset{\perp}{\oplus} L^\perp.$$

We can decompose the sequence $U_\varepsilon$ as $U_\varepsilon = \Phi_\varepsilon + v_\varepsilon$ with $\Phi_\varepsilon \in L$ and $v_\varepsilon \in L^\perp$. First, we are going to prove that $v_\varepsilon$ converges to zero using that $\mathcal{L}$ is coercive on $L^\perp$. Taking the scalar product of the equation (A.2) against $U_\varepsilon$, we find:

$$-\varepsilon |U_\varepsilon|_\mu^2 + \langle \mathcal{L} U_\varepsilon,\, U_\varepsilon\rangle_\mu = \frac{1}{N_\varepsilon}\langle \chi,\, U_\varepsilon\rangle.$$

Therefore, at the limit $\varepsilon \to 0$, we have:

$$\langle \mathcal{L} U_\varepsilon,\, U_\varepsilon\rangle_\mu \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

Since we have the equality $\langle \mathcal{L} U_\varepsilon,\, U_\varepsilon\rangle_\mu = -\alpha^2 |\partial_\kappa U_\varepsilon|_\mu^2$ (3.14) and $\partial_\kappa U_\varepsilon = \partial_\kappa v_\varepsilon$, we obtain that:

$$|\partial_\kappa v_\varepsilon|_\mu^2 \overset{\varepsilon\to 0}{\longrightarrow} 0. \tag{A.3}$$

Then we use the Poincaré inequality for Gaussian measures [26]:

$$\int_\kappa |f - \overline{f}|^2\, \mathcal{N}\, d\kappa \leq C \int_\kappa |\partial_\kappa f|^2\, \mathcal{N}\, d\kappa, \tag{A.4}$$

with $C$ a positive constant and $\overline{f}$ the mean of $f$ defined as:

$$\overline{f} = \int_\kappa f(\kappa)\, \mathcal{N}\, d\kappa.$$

Applying the Poincaré inequality (A.4) to $v_\varepsilon$ leads to:

$$\begin{aligned} |\partial_\kappa v_\varepsilon|_\mu^2 &= \int_\theta \int_\kappa |\partial_\kappa v_\varepsilon|^2 \mathcal{N}\, \mathcal{M}\, d\kappa d\theta \\ &\geq \int_\theta C^{-1} \int_\kappa |v_\varepsilon - \overline{v}_\varepsilon|^2 \mathcal{N}\, d\kappa\, \mathcal{M}\, d\theta \\ &\geq C^{-1} |v_\varepsilon - \overline{v}_\varepsilon|_\mu^2. \end{aligned} \tag{A.5}$$

Since $v_\varepsilon \in L^\perp$, for all $\Phi(\theta) \in L$, we have:

$$\int_{\theta,\kappa} v_\varepsilon(\theta,\kappa)\Phi(\theta)\mathcal{M}(\theta)\mathcal{N}(\kappa)\, d\theta d\kappa = \int_\theta \overline{v}_\varepsilon(\theta)\Phi(\theta)\mathcal{M}\, d\theta = 0.$$

Therefore $\overline{v}_\varepsilon(\theta) = 0$. Combining the inequality (A.5) with (A.3) yields:

$$|v_\varepsilon|_\mu^2 \stackrel{\varepsilon\to 0}{\longrightarrow} 0.$$

It remains to prove that $\Phi_\varepsilon$ converges to zero. With this aim, we take the scalar product of the equation (A.2) against the function $\kappa$. Once we take the limit $\varepsilon \to 0$, we find:

$$\langle \mathcal{L}U_\varepsilon,\, \kappa\rangle_\mu \stackrel{\varepsilon\to 0}{\longrightarrow} 0.$$

Using that $|\partial_\kappa v_\varepsilon|_\mu^2$ also converges to zero, we deduce that:

$$\int_{\theta,\kappa} \kappa^2 \partial_\theta U_\varepsilon \, \mu \, d\theta d\kappa \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{A.6}$$

We would like to use once again a Poincaré inequality. With this aim, we define the function $h_\varepsilon(\theta)$ as:

$$h_\varepsilon(\theta) = \int_\kappa \kappa^2 U_\varepsilon(\theta,\kappa)\, \mathcal{N}(\kappa)\, d\kappa$$

and we use the notation:

$$|h(\theta)|_{\mathcal{M}}^2 = \int_\theta |h(\theta)|^2 \, \mathcal{M}\, d\theta.$$

So equation (A.6) can be read as $|\partial_\theta h_\varepsilon|_{\mathcal{M}}^2 \stackrel{\varepsilon\to 0}{\longrightarrow} 0$. The usual Poincaré inequality gives:

$$|h_\varepsilon - \overline{h}_\varepsilon|_{\mathcal{M}}^2 \leq C |\partial_\theta h_\varepsilon|_{\mathcal{M}}^2, \tag{A.7}$$

with $\overline{h}_\varepsilon = \int_\theta h_\varepsilon(\theta)\mathcal{M}(\theta)\, d\theta$. But since we already know that $U_\varepsilon$ converges weakly to zero, we have:

$$\overline{h}_\varepsilon = \langle U_\varepsilon,\, \kappa^2\rangle_\mu \stackrel{\varepsilon\to 0}{\longrightarrow} 0.$$

Therefore the Poincaré inequality (A.7) yields $h_\varepsilon \stackrel{\varepsilon\to 0}{\longrightarrow} 0$, or in other words:

$$\int_{\theta,\kappa} \kappa^2 U_\varepsilon \, \mu \, d\theta d\kappa \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{A.8}$$

Since $v_\varepsilon$ converges to zero, equation (A.8) leads to:

$$\int_{\theta,\kappa} \Phi_\varepsilon(\theta)\kappa^2 \mathcal{M}(\theta)\mathcal{N}(\kappa)\, d\theta d\kappa \stackrel{\varepsilon\to 0}{\longrightarrow} 0,$$

which finally gives that $\Phi_\varepsilon$ also converges strongly to zero in $L^2_\mu$.
Since both $v_\varepsilon$ and $\Phi_\varepsilon$ convergence strongly to zero, $U_\varepsilon$ converges strongly to zero as well. This contradicts that $|U_\varepsilon|_\mu = 1$ for all $\varepsilon$. Therefore, the sequence $\psi_\varepsilon$ is bounded in $L^2_\mu$, so we can extract a subsequence which converges weakly to $\psi_0$ in $L^2_\mu$. This function $\psi_0$ has to satisfy:

$$\mathcal{L}\psi_0 = \chi$$

which ends the proof of the lemma.

□

# Appendix A2: Proof of lemma 3.4 (ii) (probabilistic proof)

**Proof.** The operator $\mathcal{L}$ is the infinitesimal generator of the following stochastic differential equation:

$$d\theta = \kappa dt, \tag{A.9}$$
$$d\kappa = -\lambda(\sin\theta + \kappa)\, dt + \sqrt{2}\alpha\, dB_t, \tag{A.10}$$

For any function $\varphi$ regular enough, we can define the semi-group:

$$P_t(\varphi)(\theta,\kappa) = \mathbb{E}[\varphi(X_t)|X_0 = (\theta,\kappa)],$$

with $X_t$ the stochastic process solution of (A.9)-(A.10). This defines a solution of the following equation (see [35]):

$$\begin{cases} \partial_t u = \mathcal{L}u \\ u_{t=0} = \varphi. \end{cases}$$

In particular, if we define $u(t) = P_t(\chi)$, a simple integration by part leads to:

$$u(t) - \chi = \int_0^t \mathcal{L}u(s)\, ds. \tag{A.11}$$

Therefore, we will find a solution to (3.25) if we are able to prove that $u(t) \stackrel{t\to\infty}{\longrightarrow} 0$. For that, we first notice that the equilibrium measure associated with $\mathcal{L}$ is given by $\mu$ (3.13) and its adjoint operator in $L^2_\mu$ is given by:

$$\mathcal{L}^*\psi = -\kappa\partial_\theta\psi + \lambda\sin\theta\partial_\kappa\psi - \lambda\kappa\partial_\kappa\psi + \alpha^2\partial^2_\kappa\psi.$$

Moreover, we can find a Lyapunov function associated with $\mathcal{L}$. The function $V(\theta,\kappa) = 1+\kappa^2$ satisfies:

$$\begin{aligned} \mathcal{L}^*V &= 2\lambda\sin\theta\,\kappa - 2\lambda\kappa^2 + 2\alpha^2 \\ &\leq 2\lambda\kappa - \lambda(1+\kappa^2) - \lambda\kappa^2 + \lambda + 2\alpha^2 \\ &\leq -\lambda V + 2(\alpha^2+\lambda)\mathbb{1}_{\{|\kappa|\leq 2+\sqrt{1+(2\alpha/\lambda+1)^2}\}}. \end{aligned}$$

Therefore $V$ is a Lyapunov function in the sense of [1, Def. 1.1]. Since $B = \mathbb{S}^1\times\{|\kappa| \leq 2+\sqrt{1+(2\alpha/\lambda+1)^2}\}$ is compact, $B$ is a "petite set" in the terminology [1, Def. 1.1] of Meyn & Tweedie [32]. So we can apply [1, Th. 2.1] and conclude that there exists a constant $K_2 > 0$ such that for all bounded function $\varphi$ satisfying $\int_{\theta,\kappa}\varphi\,\mu\, d\theta d\kappa = 0$, we have:

$$|P_t(\varphi)|_\mu \leq K_2|\varphi|_\infty\, \mathrm{e}^{-\lambda t}.$$

Therefore, we can pass to the limit $t \to \infty$ in (A.11) to find that:

$$-\chi = \int_0^\infty \mathcal{L}u(s)\, ds,$$

Defining the function $\psi = -\int_0^\infty u(s)\, ds$, we get a solution to:

$$\mathcal{L}\psi = \chi.$$

For the uniqueness of the solution, we proceed as in appendix A1.

□

# Appendix B: Proof of proposition 4.1.

**Proof.** We first prove the following lemma.

**Lemma A.1** *For every integer $m$ and every positive integer $n \geq 0$, we have:*

$$\mathcal{L}(\varphi_m P_n) = \sum_{\substack{-1\leq j\leq 1\\ -1\leq k\leq 1}} D^{m,n}(j,k)\varphi_{m+j}P_{n+k}$$

*with $D^{m,n}$ a $3 \times 3$ matrix given by:*

$$D^{m,n} = \begin{bmatrix} -\beta_2\sqrt{n} & 0 & \beta_2\sqrt{n+1} \\ \beta_1\, m\sqrt{n} & -\lambda n & \beta_1\, m\sqrt{n+1} \\ \beta_2\sqrt{n} & 0 & -\beta_2\sqrt{n+1} \end{bmatrix} \tag{A.12}$$

*with:*

$$\beta_1 = \frac{i\alpha}{\sqrt{\lambda}} \quad , \quad \beta_2 = \frac{i\lambda\sqrt{\lambda}}{4\alpha}.$$

**Proof.** First, using the properties of the Hermite polynomials[2], we can find several properties of $P_n$:

$$\begin{aligned} P_n' &= \frac{\sqrt{\lambda}}{\alpha}\sqrt{n}\, P_{n-1}, \\ \kappa P_n &= \frac{\alpha}{\sqrt{\lambda}}\left(\sqrt{n+1}P_{n+1} + \sqrt{n}P_{n-1}\right). \end{aligned} \tag{A.13}$$

[2] Indeed $H_n' = nH_{n-1}$ and $xH_n = H_{n+1} + nH_{n-1}$

In particular, the polynomials $P_n$ are eigenfunctions of the self-adjoint part of $\mathcal{L}$:

$$-\lambda\kappa\partial_\kappa P_n + \alpha^2\partial_k^2 P_n = -\lambda n P_n. \tag{A.14}$$

Then, we compute:

$$\mathcal{L}(\varphi_m P_n) = \kappa P_n \partial_\theta \varphi_m - \lambda \sin\theta\, \varphi_m\, \partial_\kappa P_n \;+\; \varphi_m(-\lambda\kappa\partial_\kappa P_n + \alpha^2\partial_k^2 P_n).$$

The derivative of $\varphi_m$ with respect to $\theta$ is given by:

$$\begin{aligned}
\partial_\theta\varphi_m &= \partial_\theta\left(\frac{\mathrm{e}^{im\theta}}{\sqrt{2\pi\mathcal{M}}}\right) = im\left(\frac{\mathrm{e}^{im\theta}}{\sqrt{2\pi\mathcal{M}}}\right) + \frac{\mathrm{e}^{im\theta}}{\sqrt{2\pi}}\left(-\frac{1}{2}\frac{-\frac{\lambda^2}{\alpha^2}\sin\theta\mathcal{M}}{\mathcal{M}^{3/2}}\right) \\
&= im\varphi_m + \frac{\lambda^2}{2\alpha^2}\frac{\mathrm{e}^{im\theta}}{\sqrt{2\pi}}\left(\frac{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}{2i}\,\frac{1}{\sqrt{\mathcal{M}}}\right) \\
&= im\varphi_m - \frac{i\lambda^2}{4\alpha^2}(\varphi_{m+1}-\varphi_{m-1}).
\end{aligned}$$

Using (A.13), we also have:

$$\begin{aligned}
\kappa P_n\partial_\theta\varphi_m &= \frac{\alpha}{\sqrt{\lambda}}\left(\sqrt{n+1}P_{n+1}+\sqrt{n}P_{n-1}\right)\left(im\varphi_m - \frac{i\lambda^2}{4\alpha^2}(\varphi_{m+1}-\varphi_{m-1})\right). \\
&= \frac{i\alpha}{\sqrt{\lambda}}\left(m\sqrt{n+1}P_{n+1}\varphi_m + m\sqrt{n}P_{n-1}\varphi_m\right) && \text{(A.15)} \\
&\quad -\frac{i\lambda\sqrt{\lambda}}{4\alpha}\left(\sqrt{n+1}P_{n+1}\varphi_{m+1} + \sqrt{n}P_{n-1}\varphi_{m+1}\right) && \text{(A.16)} \\
&\quad +\frac{i\lambda\sqrt{\lambda}}{4\alpha}\left(\sqrt{n+1}P_{n+1}\varphi_{m-1} + \sqrt{n}P_{n-1}\varphi_{m-1}\right). && \text{(A.17)}
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
-\lambda\sin\theta\,\varphi_m\,\partial_\kappa P_n &= -\lambda\frac{\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}}{2i}\,\varphi_m\,\frac{\sqrt{\lambda}}{\alpha}\sqrt{n}\,P_{n-1} \\
&= \frac{i\lambda\sqrt{\lambda}}{2\alpha}\sqrt{n}(\varphi_{m+1}-\varphi_{m-1})P_{n-1}. && \text{(A.18)}
\end{aligned}$$

Finally, since $P_n$ satisfies (A.14), we get:

$$\varphi_m(-\lambda\kappa\partial_\kappa P_n + \alpha^2\partial_k^2 P_n) = -\lambda n\,\varphi_m P_n. \tag{A.19}$$

Combining (A.15) (A.16) (A.17) (A.18) (A.19), we find the expression (A.12) of $D^{m,n}$. □

To find the matrix representation of the operator $\mathcal{L}$ in $V_{m,n}$, we introduce the vectors $u$ and $v$ defined by:

$$\begin{aligned} u &= (\varphi_{-m}, \, \ldots, \, \varphi_0, \, \ldots, \, \varphi_m)^T \\ v &= (P_0, \, \ldots, \, P_n)^T. \end{aligned}$$

With these notations, a function $\psi \in V_{m,n}$ with a matrix representation $X$ (4.4) can be written as:

$$\psi_{m,n} = \sum_{|j|\leq m, \, 0\leq k\leq n} C_j^k \varphi_j P_k \, = \, u^T \, X \, v.$$

Moreover, thanks to the matrices defined in (4.5), we can write for example

$$\begin{aligned} u^T \, X L_{-1} \, v &= \sum_{|j|\leq m, \, 1\leq k\leq n} C_j^k \varphi_j P_{k-1} \\ (D_1 u)^T \, X \, v &= \sum_{|j|\leq m, \, 1\leq k\leq n} j C_j^k \varphi_j P_k. \end{aligned}$$

For a function $\psi \in L^2_\mu$, using the lemma A.1, we can write:

$$\begin{aligned} \mathcal{L}\psi &= \sum_{m,n} C_m^n \left( \sum_{\substack{-1\leq j\leq 1 \\ -1\leq k\leq 1}} D^{m,n}(j,k) \varphi_{m+j} P_{n+k} \right) \\ &= \sum_{m,n} C_m^n \Big( \beta_1 m \varphi_m (\sqrt{n} P_{n-1} + \sqrt{n+1} P_{n+1}) \\ &\qquad + \beta_2 (-\varphi_{m-1} + \varphi_{m+1}) \sqrt{n} P_{n-1} \\ &\qquad\quad + \beta_2 (\varphi_{m-1} - \varphi_{m+1}) \sqrt{n+1} P_{n+1} \; - \lambda \varphi_m \, n P_n \Big). \end{aligned}$$

Therefore, for every $\varphi \in V_{m,n}$, we have:

$$\begin{aligned} \langle \mathcal{L}\psi, \, \varphi \rangle_\mu &= \langle \; \beta_1 (D_1 u)^T X (\sqrt{D_2} L_{-1} v) + \beta_1 (D_1 u)^T X (L_{+1} \sqrt{D_2} v) \\ &\quad - \beta_2 (L_{-1} u)^T X (\sqrt{D_2} L_{-1} v) + \beta_2 (L_{+1} u)^T X (\sqrt{D_2} L_{-1} v) \\ &\quad + \beta_2 (L_{-1} u)^T X (L_{+1} \sqrt{D_2} v) - \beta_2 (L_{+1} u)^T X (L_{+1} \sqrt{D_2} v) \\ &\quad - \lambda u^T X (D_2 v) \quad , \quad \varphi \rangle_\mu. \end{aligned}$$

We can simplify this expression:

$$\begin{aligned} \langle \mathcal{L}\psi, \, \varphi \rangle_\mu &= \langle \; u^T \, \beta_1 D_1 X (\sqrt{D_2} L_{-1} + L_{+1} \sqrt{D_2}) \, v \\ &\quad + u^T \, \beta_2 (-L_{-1} + L_{+1})^T X (\sqrt{D_2} L_{-1}) \, v \\ &\quad + u^T \, \beta_2 (L_{-1} - L_{+1})^T X (L_{+1} \sqrt{D_2}) \, v \\ &\quad - \lambda u^T X (D_2 v) \quad , \quad \varphi \rangle_\mu, \end{aligned}$$

which finally gives

$$\langle \mathcal{L}\psi, \varphi \rangle_\mu = \langle \, \beta_1 M_1 X N_1 \, + \, \beta_2 M_2 X N_2 \, - \, \lambda X D_2 \quad , \quad \varphi \rangle_\mu,$$

with $M_1$, $M_2$, $N_1$ and $N_2$ defined in (4.7). Therefore, using $\varphi = -\sin\theta$, we find that $X$ has to satisfy equation (4.6).

□

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