Models of flocking with asymmetric interactions

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joint work with

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Outline

1. Introduction
2. The Cucker-Smale model
   - The model
   - Flocking for the C-S model
   - Drawbacks of the C-S model
   - A new model
3. Flocking for the new model
   - $\ell^\infty$ approach
   - Active sets
   - Convergence
4. Kinetic and macroscopic equations
   - Kinetic equation
   - Macroscopic equation
   - Convergence at the macroscopic level
5. Conclusion
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   - Macroscopic equation
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5. Conclusion
What is flocking?

Nature gives many examples of flocking behavior.

There are two characteristics in a flock:

- the individuals stay at bounded distance from each other (bounded distance),
- they all move in the same direction (alignment).
Open questions

These systems ask different questions:

- How do individuals manage to create a flock?
- What kind of rules will lead to a flock?
Open questions

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- How do individuals manage to create a flock?
- What kind of rules will lead to a flock?

Those questions are difficult to answer experimentally.

*The use of models is essential.*
Boids model

Classical model with 3 zones

Boids model

## Outline

1. **Introduction**
2. **The Cucker-Smale model**
   - The model
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   - Drawbacks of the C-S model
   - A new model
3. **Flocking for the new model**
   - $\ell^\infty$ approach
   - Active sets
   - Convergence
4. **Kinetic and macroscopic equations**
   - Kinetic equation
   - Macroscopic equation
   - Convergence at the macroscopic level
5. **Conclusion**
The Vicsek model

Discrete Vicsek model (’95)

\[
\begin{align*}
    x_i^{n+1} &= x_i^n + \Delta t \omega_i^n \\
    \omega_i^{n+1} &= \overline{\Omega}_i^n + \epsilon \\
    \text{with } \overline{\Omega}_i^n &= \frac{\sum |x_j - x_i| < R \omega_j^n}{\sum |x_j - x_i| < R \omega_j^n}, \epsilon \text{ noise.}
\end{align*}
\]
The Vicsek model

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with \[ \Omega_i^n = \frac{\sum |x_j - x_i| < R \omega_j^n}{\sum |x_j - x_i| < R \omega_j^n} \], \( \epsilon \) noise.

Continuous Vicsek model ('08 Degond, M.)

\[ \frac{dx_i}{dt} = \omega_i \]

\[ d\omega_i = (\text{Id} - \omega_i \otimes \omega_i)(\nu \Omega_i dt + \sqrt{2D} dB_t) \]
The Vicsek model

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\[ \Omega_i^n = \frac{\sum |x_j - x_i| < R \omega_j^n}{\sum |x_j - x_i| < R \omega_j^n} \], \( \epsilon \) noise.

Continuous Vicsek model ('08 Degond, M.)

\[ \frac{dx_i}{dt} = \omega_i \]
\[ d\omega_i = (I_d - \omega_i \otimes \omega_i)(\nu \Omega_i dt + \sqrt{2D} dB_t) \]

Pbm: The Vicsek model is too complex to investigate analytically its asymptotic behavior.
In 2007, Cucker and Smale proposed a simplified version of the Vicsek model. They made three simplifications:

- No noise in the model ($\epsilon = 0$),
- No constraint on the velocity ($|v_i| \neq 1$),
- The mean velocity ($\bar{\Omega}_i$) is simply a sum of the other velocities, **weighted** by the distance.
The Cucker-Smale model

They end-up with the following model:

\[
\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt} = \frac{\alpha}{N} \sum_{j=1}^{N} \phi_{ij}(\mathbf{v}_j - \mathbf{v}_i), \tag{1}
\]

where \( \alpha > 0 \) and \( \phi_{ij} \) is the influence of agent \( j \) on agent \( i \):

\[\phi_{ij} := \phi(|\mathbf{x}_j - \mathbf{x}_i|).\]
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\begin{align*}
\frac{dx_i}{dt} &= v_i, \\
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\end{align*}
\] (1)

where \( \alpha > 0 \) and \( \phi_{ij} \) is the influence of agent \( j \) on agent \( i \):

\[
\phi_{ij} := \phi(|x_j - x_i|).
\]

The so-called influence function, \( \phi(\cdot) \), is a strictly positive decreasing function.

**Example:** \( \phi(r) = \frac{1}{1+r} \).
Definition of flocking

Let \( \{x_i(t), v_i(t)\} \) the positions and the velocities of \( N \) agents, and let \( d_X(t) \) and \( d_V(t) \) denote the diameters in position and velocity phase spaces:

\[
d_X(t) = \max_{i,j} |x_j(t) - x_i(t)|, \quad d_V(t) = \max_{i,j} |v_j(t) - v_i(t)|.
\]

![Diagram of flocking](image)
Definition of flocking

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\]

**Def.** The system \( \{x_i(t), v_i(t)\}_{i=1,\ldots,N} \) converges to a flock if the following two conditions hold, uniformly in \( N \),

\[
    \sup_{t \geq 0} d_X(t) < +\infty \quad \text{and} \quad \lim_{t \to +\infty} d_V(t) = 0.
\]
Flocking for the C-S model

The main result on the C-S model is the following theorem.

**Thm.** If the influence function $\phi$ decays slowly enough:

$$\int_0^\infty \phi(r) \, dr = +\infty,$$

then the C-S model converges to a flock.
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**Remarks.**
- The key tool for the proof is the *symmetry property* of the pairwise influence $\phi_{ij}$ (e.g. $\phi_{ij} = \phi_{ji}$).
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**Thm.** *If the influence function $\phi$ decays slowly enough:*

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*then the C-S model converges to a flock.*

**Remarks.**

- The key tool for the proof is the *symmetry property* of the pairwise influence $\phi_{ij}$ (e.g. $\phi_{ij} = \phi_{ji}$).
- This symmetry implies that the *total momentum* is conserved:

$$\frac{d}{dt} \left( \frac{1}{N} \sum_{i=1}^N v_i(t) \right) = 0 \quad \Rightarrow \quad \bar{v}(t) := \frac{1}{N} \sum_{i=1}^N v_i(t) = \bar{v}(0).$$
Sketch of the proof: $\ell^2$ approach

We look at the variance $Var = \frac{1}{N} \sum_i |v_i - \bar{v}|^2$:
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We look at the variance $\text{Var} = \frac{1}{N} \sum_i |v_i - \bar{v}|^2$:

$$\frac{d}{dt} \text{Var}(t) = \frac{2}{N} \sum_i \langle \dot{v}_i, v_i - \bar{v} \rangle = \frac{2}{N^2} \sum_{i,j} \phi_{ij} \langle v_j - v_i, v_i - \bar{v} \rangle$$
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$$= - \frac{1}{N^2} \sum_{i,j} \phi_{ij} |\mathbf{v}_j - \mathbf{v}_i|^2 \quad \text{(by symmetry } \phi_{ij} = \phi_{ji})$$
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$$\leq - \frac{\phi(dX)}{N^2} \sum_{i,j} |\mathbf{v}_j - \mathbf{v}_i|^2 = -2\phi(dX) \text{Var}(t).$$
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$$= -\frac{1}{N^2} \sum_{i,j} \phi_{ij} |\mathbf{v}_j - \mathbf{v}_i|^2 \quad \text{(by symmetry } \phi_{ij} = \phi_{ji})$$

$$\leq -\frac{\phi(d_X)}{N^2} \sum_{i,j} |\mathbf{v}_j - \mathbf{v}_i|^2 = -2\phi(d_X) \text{Var}(t).$$

To conclude, we use that the diameter $d_X(t)$ grows at most linearly ($d_X(t) \lesssim C \cdot t$) and the Gronwall lemma.

Ref. Cucker-Smale ('07), Ha-Tadmor ('08), Carrillo-Fornasier-Rosado-Toscani ('09), Ha-Liu ('09).
Drawbacks of the C-S model

Certain aspects of the C-S model are too simple. We mention two of them.
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1) **Symmetry of the interaction:** $\phi_{ij} = \phi_{ji}$
Certain aspects of the C-S model are too simple. We mention two of them.

1) **Symmetry of the interaction:** \( \phi_{ij} = \phi_{ji} \)

In many scenario, interaction among individual is not symmetric:

**Example 1**

*Agent j is in front of agent i.*

\[(x_i, v_i) \quad (x_j, v_j)\]
Certain aspects of the C-S model are too simple. We mention two of them.

1) **Symmetry of the interaction:** $\phi_{ij} = \phi_{ji}$

In many scenario, interaction among individual is not symmetric:

**Example 1**

Agent $j$ is in front of agent $i$.

$$(x_i, v_i) \quad (x_j, v_j)$$

**Example 2**

Agent $j$ has a lot of neighbors.

$x_i$  \quad $x_j$
Drawbacks of the C-S model

2) The weight $1/N$:

$$\frac{dv_i}{dt} = \frac{\alpha}{N} \sum_{j=1}^{N} \phi_{ij} (v_j - v_i)$$
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Example 3

In the "small" group $G_1$ alone:

$$\frac{dv_i}{dt} = \frac{\alpha}{N_1} \sum_{j=1}^{N_1} \phi_{ij}(v_j - v_i)$$
**Drawbacks of the C-S model**

2) The weight $1/N$:

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**Example 3**

In the “small” group $G_1$ with the “large” group $G_2$:

$$\frac{dv_i}{dt} = \frac{\alpha}{N_1 + N_2} \sum_{j=1}^{N_1+N_2} \phi_{ij}(v_j - v_i)$$
Drawbacks of the C-S model

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Example 3

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\[
\frac{dv_i}{dt} = \frac{\alpha}{N_1 + N_2} \sum_{j=1}^{N_1+N_2} \phi_{ij}(v_j - v_i) \approx \frac{\alpha}{N_1} \sum_{j=1}^{N_1} \phi_{ij}(v_j - v_i)
\]
Drawbacks of the C-S model

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In the “small” group $G_1$ with the “large” group $G_2$:

$$\frac{dv_i}{dt} = \frac{\alpha}{N_1 + N_2} \sum_{j=1}^{N_1 + N_2} \phi_{ij}(v_j - v_i) \approx \frac{\alpha}{N_1 + N_2} \sum_{j=1}^{N_1} \phi_{ij}(v_j - v_i) \approx 0!$$
A new model

We propose the following dynamical system:

\[
\begin{align*}
\frac{d\mathbf{x}_i}{dt} &= \mathbf{v}_i, \\
\frac{d\mathbf{v}_i}{dt} &= \frac{\alpha}{\sum_{k=1}^{N} \phi_{ik}} \sum_{j=1}^{N} \phi_{ij} (\mathbf{v}_j - \mathbf{v}_i),
\end{align*}
\]

(2)

with \( \phi_{ij} = \phi(|\mathbf{x}_j - \mathbf{x}_i|) \) and \( \alpha > 0 \).
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with \( \phi_{ij} = \phi(|x_j - x_i|) \) and \( \alpha > 0 \).

The influence of the agent \( j \) on agent \( i \) is weighted by the total influence, \( \sum_{k=1}^{N} \phi_{ik} \), exerted on agent \( i \).
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**Remark.** If \( \phi_{ij} \approx \phi_0 \) \( \Rightarrow \) the C-S dynamics.
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\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \frac{\alpha}{\sum_{k=1}^{N} \phi_{ik}} \sum_{j=1}^{N} \phi_{ij} (v_j - v_i),
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with \( \phi_{ij} = \phi(|x_j - x_i|) \) and \( \alpha > 0 \).

The influence of the agent \( j \) on agent \( i \) is weighted by the total influence, \( \sum_{k=1}^{N} \phi_{ik} \), exerted on agent \( i \).

**Remark.** If \( \phi_{ij} \approx \phi_0 \Rightarrow \) the C-S dynamics. Otherwise the model better captures strongly “non-homogeneous” scenarios (Example 3).
A new model

The model can be written as:

\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \\
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A new model

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\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i),
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with:

\[
a_{ij} := \frac{\phi(|x_j - x_i|)}{\sum_{k=1}^{N} \phi(|x_k - x_i|)} \geq 0, \quad \sum_{j} a_{ij} = 1.
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\]

The new model lacks the **symmetry property**:

Non-symmetric interaction

\[
a_{ij} \neq a_{ji}
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A new model

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\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \alpha \sum_{j=1}^{N} a_{ij} (v_j - v_i),
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The new model lacks the symmetry property:

Non-symmetric interaction

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a_{ij} \neq a_{ji}
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The total momentum \((\bar{v} = \frac{1}{N} \sum_i v_i)\) is not preserved in the model!
$\ell^\infty$ approach

\[
\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \alpha \sum_{j=1}^{N} a_{ij} (v_j - v_i), \quad \text{with} \quad a_{ij} \geq 0, \quad \sum_{j} a_{ij} = 1.
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\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \alpha \sum_{j=1}^{N} a_{ij} (v_j - v_i), \quad \text{with} \quad a_{ij} \geq 0, \quad \sum_j a_{ij} = 1.
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- Here, the total momentum \( \bar{v} \) is not preserved, the variance \( \text{Var} \) may increase.
The Cucker-Smale model

Flocking for the new model

Kinetic and macroscopic

Conclusion

\[ \ell^\infty \text{ approach} \]

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\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \alpha \sum_{j=1}^{N} a_{ij} (v_j - v_i), \quad \text{with} \quad a_{ij} \geq 0, \quad \sum_j a_{ij} = 1. 
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- Here, the total momentum \( \bar{v} \) is not preserved, the variance \( \Var \) may increase.
- Instead, we investigate the evolution of the diameter \( dV \):

\[ dX \]

\[ dX \]

Sébastien Motsch (CSCAMM)
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\ell^\infty \text{ approach}

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\end{align*}
\]

Here, the total momentum $\bar{v}$ is not preserved, the variance $Var$ may increase.

Instead, we investigate the evolution of the diameter $d_V$:

\[
\begin{align*}
\frac{dX}{dt} &\quad \text{and} \\
\frac{dX_p}{dt} &\quad \text{and} \\
\frac{dX_q}{dt} &\quad \text{and}
\end{align*}
\]
\( \ell^\infty \) approach

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\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \alpha \sum_{j=1}^{N} a_{ij}(v_j - v_i), \quad \text{with} \quad a_{ij} \geq 0, \quad \sum_j a_{ij} = 1.
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\[ d_X \]

\[ d_V \]

\[ v_p \]

\[ v_q \]
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\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \alpha \sum_{j=1}^{N} a_{ij} (v_j - v_i), \quad \text{with} \quad a_{ij} \geq 0, \quad \sum_j a_{ij} = 1.
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\begin{itemize}
  \item Here, the total momentum \( \bar{v} \) is not preserved, the variance \( \text{Var} \) may increase.
  \item Instead, we investigate the evolution of the diameter \( d_V \):
\end{itemize}

\[ \Omega := \text{Conv}\{v_1, \ldots, v_N\} \]

\[ d_V = \text{dist}(v_p, v_q) \]

\[ d_X = \text{dist}(x_p, x_q) \]
Is $d_V$ decreasing?
**Is \( d_V \) decreasing?**

**Trick:** \( \dot{v}_p = \alpha \sum_j a_{pj} (v_j - v_p) \)
Is $d_V$ decreasing?

**Trick:** $\dot{v}_p = \alpha (\bar{v}_p - v_p)$ with $\bar{v}_p = \sum_j a_{pj} v_j$
Is $d_V$ decreasing?

**Trick:** $\dot{v}_p = \alpha (\bar{v}_p - v_p)$ with $\bar{v}_p = \sum_j a_{pj} v_j \in \Omega$
**Is \( \text{d}_V \) decreasing?**

**Trick:** \( \dot{v}_p = \alpha(\overline{v}_p - v_p) \) with \( \overline{v}_p = \sum_j a_{pj}v_j \in \Omega \)

Therefore,

\[
\frac{d}{dt} |v_p - v_q|^2
\]
**Is $d_V$ decreasing?**

**Trick:** $\dot{v}_p = \alpha (\bar{v}_p - v_p)$ with $\bar{v}_p = \sum_j a_{pj} v_j \in \Omega$

Therefore,

$$\frac{d}{dt} |v_p - v_q|^2 = 2 \langle \dot{v}_p - \dot{v}_q, v_p - v_q \rangle$$

$$= 2 (\langle \bar{v}_p - \bar{v}_q, v_p - v_q \rangle - |v_p - v_q|^2)$$

$$\leq 2 |v_p - v_q| (|\bar{v}_p - \bar{v}_q| - |v_p - v_q|) \leq 0.$$
Is $d_V$ decreasing?

**Trick:** $\dot{v}_p = \alpha(\bar{v}_p - v_p)$ with $\bar{v}_p = \sum_j a_{pj}v_j \in \Omega$

Therefore,

$$\frac{d}{dt}|v_p - v_q|^2 = 2\langle \dot{v}_p - \dot{v}_q, v_p - v_q \rangle$$

$$= 2\left(\langle \bar{v}_p - \bar{v}_q, v_p - v_q \rangle - |v_p - v_q|^2 \right)$$

$$\leq 2|v_p - v_q|\left(|\bar{v}_p - \bar{v}_q| - |v_p - v_q| \right) \leq 0.$$ 

So $d_V(t)$ is decreasing in time.
Is $d_V$ decreasing?

**Trick:** $\dot{v}_p = \alpha ( \bar{v}_p - v_p )$ with $\bar{v}_p = \sum_j a_{pj}v_j \in \Omega$

Therefore,

$$\frac{d}{dt} \left| v_p - v_q \right|^2 = 2 \langle \dot{v}_p - \dot{v}_q, v_p - v_q \rangle$$

$$= 2 \left( \langle \bar{v}_p - \bar{v}_q, v_p - v_q \rangle - |v_p - v_q|^2 \right)$$

$$\leq 2 |v_p - v_q| \left( |\bar{v}_p - \bar{v}_q| - |v_p - v_q| \right) \leq 0.$$

So $d_V(t)$ is **decreasing** in time. *By how much?*
Active sets

Let \( \{a_{ij}\} \) be a normalized influence matrix, \( a_{ij} > 0, \sum_j a_{ij} = 1 \).
Active sets

Let \( \{a_{ij}\} \) be a normalized influence matrix, \( a_{ij} > 0, \sum_j a_{ij} = 1. \)

**Def.** The active set, \( \Lambda_p(\theta) \), is the set of agents which influence “\( p \)” more than \( \theta \),

\[
\Lambda_p(\theta) := \{j \mid a_{pj} \geq \theta\}.
\]

The global active set, \( \Lambda(\theta) \), is the intersection of all the active sets at that level,

\[
\Lambda(\theta) = \bigcap_p \Lambda_p(\theta).
\]
Active sets

Let \( \{a_{ij}\} \) be a normalized influence matrix, \( a_{ij} > 0, \sum_j a_{ij} = 1. \)

**Def.** The active set, \( \Lambda_p(\theta) \), is the set of agents which influence “\( p \)” more than \( \theta \),

\[
\Lambda_p(\theta) := \{j \mid a_{pj} \geq \theta\}.
\]

The global active set, \( \Lambda(\theta) \), is the intersection of all the active sets at that level,

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**Remark.** This notion of active set, \( \Lambda_p(\theta) \), defines a “neighborhood” for agent “\( p \)”.

---

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Key lemma

**Lemma.** Let \( S \) be an *antisymmetric* matrix bounded by \( M \), \( u, w \) be two positive vectors \((u_i, w_i \geq 0)\) satisfying \( \sum_i u_i = \sum_j w_j = 1 \). Then,

\[
| \sum_{i,j} S_{ij} u_i w_j | \leq M
\]
**Key lemma**

**Lemma.** Let $S$ be an *antisymmetric* matrix bounded by $M$, $u$, $w$ be two positive vectors $(u_i, w_i \geq 0)$ satisfying $\sum_i u_i = \sum_j w_j = 1$. Then, for every $\theta > 0$, we have

$$| \sum_{i,j} S_{ij} u_i w_j | \leq M \left( 1 - \lambda^2(\theta) \theta^2 \right),$$

where $\lambda(\theta)$ denotes the number of “active entries”

$$\lambda(\theta) := \# \{ j \mid u_j \geq \theta \text{ and } w_j \geq \theta \}.$$
Key lemma

Proof lemma.

\[
\sum_{i,j} S_{ij} u_i w_j = \frac{1}{2} \sum_{i,j} S_{ij} (u_i w_j - u_j w_i) \leq \frac{M}{2} \sum_{i,j} |u_i w_j - u_j w_i|.
\]
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Proof lemma.

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The identity \(|a - b| \equiv a + b - 2 \min(a, b)\) for \(a, b \geq 0\) yields:

\[ \sum_{i,j} S_{ij} u_i w_j \leq \frac{M}{2} \left( \sum_{i,j} u_i w_j + u_j w_i - 2 \min\{u_i w_j, u_j w_i\} \right) \]
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\[ = M \left( 1 - \sum_{i,j} \min\{u_i w_j, u_j w_i\} \right). \]

Restricting the sum only to the pairs of active entries ends the proof.
Proposition

Fix $\theta > 0$ and let $\lambda(\theta)$ be the number of agents in the global active set, $\Lambda(\theta)$. Then the diameters $d_X(t)$ and $d_V(t)$ satisfy,

$$\frac{d}{dt} d_X(t) \leq d_V(t)$$
$$\frac{d}{dt} d_V(t) \leq -\alpha \lambda^2(\theta) \theta^2 d_V(t).$$
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$$

**Proof.** Remember:

$$
\frac{d}{dt} |\mathbf{v}_p - \mathbf{v}_q|^2 \leq 2 |\mathbf{v}_p - \mathbf{v}_q| (|\overline{\mathbf{v}}_p - \overline{\mathbf{v}}_q| - |\mathbf{v}_p - \mathbf{v}_q|)
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\]

But:

\[
\bar{\mathbf{v}}_p - \bar{\mathbf{v}}_q = \sum_j a_{pj} \mathbf{v}_j - \bar{\mathbf{v}}_q = \sum_j a_{pj} (\mathbf{v}_j - \bar{\mathbf{v}}_q)
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Proof. Remember:

$$\frac{d}{dt} |v_p - v_q|^2 \leq 2|v_p - v_q|(|v_p - \bar{v}_q| - |v_p - v_q|)$$

But:

$$\bar{v}_p - \bar{v}_q = \sum_j a_{pj} v_j - \bar{v}_q = \sum_j a_{pj} (v_j - \bar{v}_q)$$
$$= \sum_{i,j} a_{pj} a_{qi} (v_j - v_i) = \sum_{i,j} u_i w_j S_{ij}.$$
Applying the key lemma with $|S_{ij}| \leq |v_p - v_q|$ leads to:

$$\bar{v}_p - \bar{v}_q \leq |v_p - v_q|(1 - \lambda_{pq}(\theta)^2\theta^2),$$
Applying the key lemma with $|S_{ij}| \leq |\mathbf{v}_p - \mathbf{v}_q|$ leads to:

$$\mathbf{v}_p - \mathbf{v}_q \leq |\mathbf{v}_p - \mathbf{v}_q|(1 - \lambda_{pq}(\theta)^2 \theta^2),$$

for any $\theta > 0$ with $\lambda_{pq}(\theta)$ the number of “active entries”:

$$\lambda_{pq}(\theta) = \#\{j \mid a_{pj} \geq \theta \text{ and } a_{qj} \geq \theta\}.$$
Applying the key lemma with \( |S_{ij}| \leq |v_p - v_q| \) leads to:

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\]

Therefore,

\[
\frac{d}{dt} |v_p - v_q|^2 \leq 2|v_p - v_q| (|\bar{v}_p - \bar{v}_q| - |v_p - v_q|)
\]

\[
\leq -2|v_p - v_q| \lambda_{pq}(\theta)^2 \theta^2 |v_p - v_q|.
\]

\(\square\)
Applying the key lemma with $|S_{ij}| \leq |v_p - v_q|$ leads to:

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Therefore,

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□

To conclude we need to find an appropriate $\theta$ for which we can count the number of active entries $\lambda_{pq}(\theta)$. 
Theorem [M,Tadmor,2011]

If the influence function $\phi$ decays slowly enough:

$$
\int_0^{\infty} \phi^2(r) \, dr = +\infty,
$$

then the new model converges to a flock.
Theorem [M, Tadmor, 2011]

If the influence function $\phi$ decays slowly enough:

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Proof. 1) Take $\theta = \frac{\phi(dx)}{N}$. We have: $a_{ij} = \frac{\phi_{ij}}{\sum_k \phi_{ik}} \geq \frac{\phi(dx)}{N} = \theta$. 

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Thus, \( \lambda(\theta) = N \). Applying the proposition gives:

\[
\dot{d_X}(t) \leq d_V(t) \quad , \quad \dot{d_V}(t) \leq -\alpha \phi^2(d_X(t)) \, d_V(t).
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$$\dot{dX}(t) \leq dV(t), \quad \dot{dV}(t) \leq -\alpha \phi^2(dX(t)) \, dV(t).$$

2) Using $\mathcal{E}(dX,dV)(t) := dV(t) + \alpha \int_0^{dX(t)} \phi^2(s) \, ds$ [Ha-Liu]:

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Theorem [M, Tadmor, 2011]

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then the new model converges to a flock.

**Proof.** 1) Take $\theta = \Phi(dX) / N$. We have: $a_{ij} = \frac{\phi_{ij}}{\sum_k \phi_{ik}} \geq \frac{\phi(dX)}{N} = \theta$. Thus, $\lambda(\theta) = N$. Applying the proposition gives:

$$\dot{d}_X(t) \leq d_V(t), \quad d_V(t) \leq -\alpha \phi^2(d_X(t)) \, d_V(t).$$

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$\Rightarrow d_V(t) \to 0$ expo. fast.
Outline

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Kinetic equation

Starting from the dynamical system,

\[
\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \frac{\alpha}{\sum_{k=1}^{N} \phi_{ik}} \sum_{j=1}^{N} \phi_{ij} (v_j - v_i),
\]
Kinetic equation

Starting from the dynamical system,

\[ \frac{dx_i}{dt} = v_i, \quad \frac{d\mathbf{v}_i}{dt} = \frac{\alpha}{\sum_{k=1}^{N} \phi_{ik}} \sum_{j=1}^{N} \phi_{ij} (\mathbf{v}_j - \mathbf{v}_i), \]

we deduce that the distribution of particles \( f(t, x, v) \) satisfies:

\[ \partial_t f + \mathbf{v} \cdot \nabla_x f + \nabla_v \cdot (F[f] f) = 0, \quad (3) \]
Starting from the dynamical system,

\[
\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i}{dt} = \frac{\alpha}{\sum_{k=1}^{N} \phi_{ik}} \sum_{j=1}^{N} \phi_{ij} (\mathbf{v}_j - \mathbf{v}_i),
\]

we deduce that the distribution of particles \( f(t, \mathbf{x}, \mathbf{v}) \) satisfies:

\[
\partial_t f + \mathbf{v} \cdot \nabla_x f + \nabla_v \cdot (F[f] f) = 0,
\]

where the vector field \( F[f] \) is given by,

\[
F[f](\mathbf{x}, \mathbf{v}) := \alpha \frac{\int_{y,w} \phi(|y-x|) (w-v)f(y,w) \, dy \, dw}{\int_y \phi(|y-x|) \rho(y) \, dy}
\]

with \( \rho(y) = \int_w f(y,w) \, dw \).
Starting from the dynamical system,

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = \frac{\alpha}{\sum_{k=1}^{N} \phi_{ik}} \sum_{j=1}^{N} \phi_{ij} (v_j - v_i),$$

we deduce that the distribution of particles $f(t, x, v)$ satisfies:

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F[f] f) = 0,$$

where the vector field $F[f]$ is given by,

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with $\rho(y) = \int_w f(y, w) dw$.

Remark. To derive rigorously the kinetic equation, once has to be careful that $F[f]$ is not lipschitz [Boissard,M. work in preparation].
Macroscopic equations

Integrating (3) against $1$ and $v$ yields:

$$\partial_t \rho + \nabla_x \cdot (\rho u) = 0 \quad \tag{4}$$

$$\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + P) = S(u), \quad \tag{5}$$
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$$S(u)(x) = \alpha \frac{\int_y \phi(|y-x|) \rho(x) \rho(y) \left(u(y) - u(x)\right) dy}{\int_y \phi(|y-x|) \rho(y) dy}.$$

and $P$ involving moment of order 2 of $f$. 
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\end{align*}
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S(u)(x) = \alpha \frac{\int_y \phi(|y-x|)\rho(x)\rho(y)(u(y) - u(x)) \, dy}{\int_y \phi(|y-x|)\rho(y) \, dy}.
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and \( P \) involving moment of order 2 of \( f \).

To close the system, we suppose that \( f \) is a monophase distribution:

\[
f(x, v) = \rho(x)\delta_{u(x)}.
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Under this assumption, $P = 0$. 
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**Hypothesis:** Let \((\rho_0, u_0)\) an initial data compactly supported. We assume that the system (4,5) admits a unique smooth solution \((\rho(t), u(t))\) for all \(t \geq 0\).
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**Theorem [M,Tadmor,2011]**

If the influence function \(\phi\) decays slowly enough:

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\int_0^\infty \phi^2(r) \, dr = +\infty,
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then the system (4,5) converges to a flock in the sense that:

\[
d_X(t) := \sup\{|x - y|, \ x, y \in \text{Supp}(\rho(t))\} \quad \text{is bounded}
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\[
d_V(t) := \sup\{|u(t, x) - u(t, y)|, \ x, y \in \text{Supp}(\rho(t))\} \quad \xrightarrow{t \to \infty} 0.
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\]

**Proof.** We use that the decay of \(d_V\) at the particle level is independent of \(N\).
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- In contrast with the C-S model, we do NOT what will be the asymptotic velocity of the flock $v_\infty$:
  \[ \Rightarrow v_\infty \text{ 'emerges' from the dynamics.} \]
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- The condition for flocking is more restrictive than the C-S model: $\int \phi(r)^2 \, dr = \infty$ instead of $\int \phi(r) \, dr = \infty$ for the C-S model.
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- Is there a flock when we only have: $\int \phi(r) dr = \infty$?
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- Flocking with $\phi$ \textit{compactly supported}...