Ex 1. [3pts] Let \( f(x) = xe^x - 1 \).

a) To apply the bisection method, we need to find \( a, b \) such that \( f(a)f(b) < 0 \). Here, we can choose: \( a = 0 \) \((f(a) = -1 < 0)\), \( b = 1 \) \((f(b) \approx 1.72 > 0)\). For the Newton’s method, we have to start “close” to a zero of \( f \). Here, we can try with \( x_0 = 0 \).

b) The error of both method is represented in the following figure ??:

![Graph showing error of Newton and bisection methods](image)

Figure 1: Evolution of the error \(|x_n - x_*|\) at each iteration for the Newton’s method (blue) and the bisection method (green) in log scale in \( y \). In this scale, the convergence of the Newton’s method is quadratic and the bisection method is only linear.

b*) To estimate the decay, we denote \( e_n = |x_n - x_*| \) where \( x_n \) is given by the algorithm (Newton or bisection) and \( x_* \) (the zero of \( f \)) is taken as the last estimate of the Newton’s method.

To measure the convergence, we need to analyze the sequence \( \{e_n\} \) in log scale:

\[
y_n = \ln e_n.
\]

Then using a linear regression (\texttt{polyfit(1:N,y,1)} in Octave/Matlab), we obtain for the bisection method (using 50 points):

\[
y_n \approx -0.693n - 1.73.
\]
Thus, $e_n \approx 0.18 e^{-0.69n} = 0.18(0.50018)^n$. This result was expected since $|x_n - x_*| \approx C(\frac{1}{2})^n$.

For the Newton’s method, it is more delicate. First, we use only 7 points since for $n > 6, x_n$ is ‘numerically’ equal to zero. Then, looking at the figure ?, the behavior of $y_n$ seems quadratic. Thus, we do a regression with a polynomial of order 2:

$$y_n \approx -1.59 \cdot n^2 + 4.13 \cdot n - 2.13.$$ 

Thus, $e_n \approx 0.12 \cdot (62)^n \cdot (0.2)^n$.

Ex 2. 4pts

Let $\varphi(x) = \sqrt{x + 1}$.

a) We show that $\varphi$ is a contraction on $I = [0, \infty)$.

\begin{itemize}
  \item[$\circ$] For any $x \geq 0$, $\varphi(x) \geq 0$. Thus, $\varphi(I) \subset I$.
  \item[$\circ$] $|\varphi'(x)| = \left|\frac{1}{2(x+1)}\right| \leq \frac{1}{2}$ on $I$.

Thus, for any $x, y \in I$, we have: $|\varphi(x) - \varphi(y)| \leq \frac{1}{2}|x - y|$.
\end{itemize}

Therefore, $\varphi$ is a contraction on $I$. We deduce that there exists a unique fixed point $x_*$ of $\varphi$ on $I$.

b) Let $J = [1, 2]$. We have $\varphi(J) \subset J$, thus we can narrow our interval ($x_* \in J$).

Consider the sequence $\{x_n\}_n$ defined recursively by: $x_{n+1} = \varphi(x_n)$ with $x_0 = 1$. We have:

$$|x_n - x_*| \leq \frac{k^n}{1-k} \cdot |b-a| = 1 \cdot \left(\frac{1}{2}\right)^{n-1}.$$ 

Thus, in order to have the error $e_n = |x_n - x_*|$ less than $10^{-4}$, a sufficient condition is:

$$\frac{1}{2^{n-1}} \leq 10^{-4} \Rightarrow n \geq \frac{4 \ln 10}{\ln 2} + 1 \approx 14.3.$$ 

After 15 iterations, the error is less than $10^{-4}$.

Ex 3. See the script on the last page.

Ex 4. Let $\varphi(x) = \sqrt{x^2 + 1}$ and $I = [0, \infty)$.

a) By the mean value theorem:

$$|\varphi(x) - \varphi(y)| = |\varphi'(c)(x - y)|,$$

with $c \in [x, y]$. Since $|\varphi'(s)| = \left|\frac{s}{\sqrt{s^2+1}}\right| < 1$, we deduce that: $|\varphi(x) - \varphi(y)| < |x - y|$.

b) We notice that $x_n = \sqrt{n}$. Thus, the sequence does not converge.
c) $\varphi(x) = x$ implies: $\sqrt{x^2 + 1} = x$ which is not possible. Thus, $\varphi$ does not have a fixed point.

This does not contradict the fixed-point theorem since $\varphi$ is not a contraction: it does not exist $k < 1$ such that $|\varphi(x) - \varphi(y)| \leq k|x - y|$.

Ex 5. [3pts]
Let $f(x) = e^x$. Consider the 3-midpoint formula to estimate $f'(x)$:

$$f'(x) \approx \frac{f(x + h) - f(x - h)}{2h}$$

a) The estimation of $f'(2)$ using the 3-midpoint formula gives:

<table>
<thead>
<tr>
<th>$h$</th>
<th>estimation $f'(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}$</td>
<td>7.40138</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>7.38918</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>7.38906</td>
</tr>
</tbody>
</table>

b) We have to find an upper-bound for $f^{(3)}(x)$ on $[x_0 - h, x_0 + h]$:

$$|f^{(3)}(\xi)| \leq e^{2+h} \leq e^{2+.1} \approx 8.2,$$

since $|h| \leq .1$. Therefore,

$$\left| f'(x) - \frac{f(x + h) - f(x - h)}{2h} \right| \leq \frac{8.2}{6}h^2.$$

.5pt

.5pt

c) We estimate the error of the 3-midpoint formula and the upper bound we have found in b)

<table>
<thead>
<tr>
<th>$h$</th>
<th>error</th>
<th>max. error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-1}$</td>
<td>$1.23 \cdot 10^{-2}$</td>
<td>$1.37 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>$10^{-2}$</td>
<td>$1.23 \cdot 10^{-4}$</td>
<td>$1.37 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>$10^{-3}$</td>
<td>$1.23 \cdot 10^{-6}$</td>
<td>$1.37 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>
%%% Fixed-point method %%%

%%% function phi
phi = @(x) exp(-x);

%%% numerical parameters
N = 10; % number iteration
x0 = 1; % initial value

%%% Saving
saveX = zeros(1,N+1);

%---------------- Loop ----------------
%----------------

%%% initialization
x = x0;
saveX(1) = x;

%%% loop
for i=1:N
    % x^n+1 = phi(x^n)
    x = phi(x);
    % save
    saveX(i+1) = x;
end

%%% Estimation error: x* = last value of x_n
xS = saveX(end);
errorFixedPointMethod = sqrt( (saveX-xS).^2);

%---------------- plot
figure(1);
clf
xInt = 0:.01:5;

%%% trick
xSpiral = [saveX' saveX']'(:,);
ySpiral = [0; phi(xSpiral(1:(end-1))));

%axis([-2 2 -2 2],'equal')
xlabel('x')
title('Fixed-point method')
legend(['f(x) = exp(-x)', 'y=x', 'location','northwest'])
axis([0 1.5 0 1.5])
figure(2);
clf
semilogy(0:(N-1),errorFixedPointMethod(1:N),'-or','linewidth',3)
xlabel('number of iterations (n)')
ylabel('error: |x_n-x_*|')
title('Error of the fixed-point method')
print('error_fixed-point_semilog.eps','-depsc2')