Ex 1. [Mid-point method]
We consider the Runge-Kutta method given by the following Butcher table:

\[
\begin{array}{c|cc}
0 & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 1 & 0
\end{array}
\]

a) Write the numerical scheme associated with this method (i.e. \( k_1 = f(t_n, y_n), k_2 = \ldots \)).

b) Show that the method is of order 2: if \( y_n = y(t_n) \), then \( y_{n+1} = y(t_{n+1}) + O(\Delta t^3) \).

Hint: Write \( y_{n+1} = y_n + f(t_n + \Delta t/2, \ldots) \) and use Taylor expansion.

Ex 2. [Numerical exploration]

a) Implement the improved Euler method:

\[
\begin{array}{c|cc}
0 & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 1 & \frac{1}{2}
\end{array}
\]

b) Compare this scheme with the Euler method for an ODE of your choice.

Hint: plot the errors \(|y(t_n) - y_n|\) at \( t_n = 1 \) for different choices of \( \Delta t \) with \( y_n \) given by the Euler method and improved Euler method.

Ex 3. Use the Euler and improved Euler method to develop an error-control method.

**Algorithm:** given \((t_n, y_n)\)

1. Estimate \( y_{n+1} \) (Euler method) and \( \tilde{y}_{n+1} \) (improved Euler method).

2. Let \( q = \left| \frac{\varepsilon \cdot \Delta t}{y_{n+1} - \tilde{y}_{n+1}} \right| \).

   - If \( q \geq 1 \) then keep \( \tilde{y}_{n+1} \) and update \( \tilde{\Delta t} = \frac{q}{2} \Delta t \).
   - Otherwise (\( q < 1 \)), let \( \tilde{\Delta t} = \frac{\Delta t}{2} \) and go back to 1.

**Application:** Use the method to solve

\[
\begin{align*}
y' &= \frac{\cos(y-t)}{10^{-3} + \cos^2 t} \\
y(0) &= 0
\end{align*}
\]

for \( 0 \leq t \leq 5 \), with the tolerance \( \varepsilon = 10^{-2} \) and an initial time step \( \Delta t = 10^{-1} \).

Compare with the solution given by the Euler method.