Ex 1.

a) Consider the boundary-value problem:

\[ \begin{cases} y'' + 5y = 0 \\ y(0) = 0, \ y(b) = \beta. \end{cases} \]

Find \( b \) and \( \beta \) such that the problem has:

i) no solution  
ii) only one solution  
iii) infinitely many solutions.

*Hint: find the general solution of \( y'' + 5y = 0 \).*

b) Consider now the boundary-value problem:

\[ \begin{cases} y'' - 5y = 0 \\ y(0) = 0, \ y(b) = \beta. \end{cases} \]

What happens here?

*Hint: use the theorem of existence and uniqueness of solution for BVP.*

Ex 2. Implement the linear shooting-method to solve:

\[ \begin{cases} y'' = p(x)y' + q(x)y + r(x) \\ y(a) = \alpha, \ y(b) = \beta. \end{cases} \]

**Application.** Find the numerical solution of:

\[ \begin{cases} y'' = 4(y - x) \\ y(0) = 0, \ y(1) = 2. \end{cases} \]

[Extra] Find the accuracy of the scheme.

*Hint: the exact solution is \( y(x) = \frac{e^{2x}}{e^{2x} - e^{-2x}} + x \).*

Ex 3. Consider the non-linear BVP:

\[ \begin{cases} y'' = y^3 - yy' \\ y(1) = \frac{1}{2}, \ y(2) = \frac{1}{3}. \end{cases} \]

Use the non-linear shooting method [with the secant method] to find the solution with tolerance \( 10^{-3} \).

Compare the solution with the explicit solution: \( y(x) = \frac{1}{1 + x} \).
Review

**Algorithm linear shooting:** Fix $\Delta x = \frac{b-a}{N}$. Use RK4 to solve the two initial value problems:

\[ i) \begin{cases} 
  y'' = p(x)y' + q(x)y + r(x) \\
  y(a) = \alpha \\
  y'(a) = 0.
\end{cases} \quad ii) \begin{cases} 
  y'' = p(x)y' + q(x)y + 0 \\
  y(a) = \alpha \\
  y'(a) = 1.
\end{cases} \]

Denote by $y_1$ and $y_2$ (resp.) the solutions. The solution of the BVP is given by the linear combination:

\[ y(x) = y_1(x) + \beta - y_1(b) \frac{y_2(b)}{y_2(b)} y_2(x). \]

**Algorithm non-linear shooting:**

1) Fix $\Delta x = \frac{b-a}{N}$, a tolerance $TOL$ and two values for $s_0$ and $s_1$ (ex. $s_0 = 0$ and $s_1 = 1$).

2) Use RK4 to solve the two initial value problems:

\[ i) \begin{cases} 
  y'' = f(x, y, y') \\
  y(a) = \alpha \\
  y'(a) = s_0.
\end{cases} \quad ii) \begin{cases} 
  y'' = f(x, y, y') \\
  y(a) = \alpha \\
  y'(a) = s_1.
\end{cases} \]

Denote the solutions by (resp.) $y(x; s_0)$ and $y(x; s_1)$.

3) If $|y(b; s_1) - \beta| < TOL$ then STOP: $y(x; s_1)$ is acceptable for the solution.

Else [correction shooting angle] define:

\[ s_2 = s_1 - \frac{y(b; s_1) - \beta}{y(b; s_1) - y(b; s_0)} (s_1 - s_0). \]

Go back to 2) with $(s_1, s_2)$ (no need to solve again for $y(x; s_1)$).