Exercise 1. Let \( f(x) \) a smooth function. We consider the backward difference method 
\[ P_1(h) = \frac{f(x^*) - f(x^* - h)}{h}. \]

a) Show that the method is 1st-order accurate: 
\[ |f'(x^*) - P_1(h)| = O(h). \]

b) Use the Richardson’s method to build a more accurate method \( P \) based on \( P_1 \).
Find the accuracy of the new method \( P \).

Exercise 2. Consider the differential equation:
\[ \begin{cases} 
  y' = \cos y - \sin y \\
  y(0) = 0 
\end{cases} \]

a) Show there exists a unique solution \( y(t) \) defined for all time \( t \geq 0 \).
Find an upper and lower bound for \( y(t) \) for \( t \geq 0 \).

b) Suppose we want to use the Euler method to estimate \( y(2) \).
Determine a time step \( \Delta t \) such that the estimation is with accuracy \( 10^{-3} \).
Hint: \[ |y(t_n) - y_n| \leq \frac{M \Delta t}{2L} (e^{L(t_n)} - 1) \] with \( M = \max_{[0,T]} |y''| \) and \( L = \max_{D} |\partial_y f| \) with \( D = [0, T] \times \mathbb{R} \).

c) Find the equilibria and study their stability.

Extra) Determine the behavior of the solution \( y(t) \) as \( t \to +\infty \).

Exercise 3. Consider the Runge-Kutta method given by the Butcher tableau:
\[
\begin{array}{c|cc}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & 1 \\
\end{array}
\]

a) Write the numerical scheme associated with this method.

b) Show that the numerical scheme is 2nd-order accurate, i.e.
\[ \text{if } y_n = y(t_n) \text{ then } y_{n+1} = y(t_{n+1}) + O(\Delta t^3). \]

Exercise 4. We consider the ODE: \( x'' = \frac{2x}{1 + x^2} \).

a) Write the ODE as a 1st order system.
Find the energy associated with the ODE.

b) Implement the Improved Euler method.

Extra) Study the stability of the equilibrium: \((x^*, y^*) = (0, 0)\).