Exercise 1. 25pts

a) We use a Taylor expansion of \( f \) of order 1:

\[
f(x_\ast - h) = f(x_\ast) - f'(x_\ast)h + O(h^2).
\]

Thus, we obtain

\[
P_1(h) = \frac{f'(x_\ast)h + O(h^2)}{h} = f'(x_\ast) + O(h).
\]

Thus, the method is 1st-order accurate.

b) Doing a Taylor expansion of \( P_1 \), we find:

\[
P_1(h) = f'(x_\ast) + Kh + O(h^2)
\]

\[
P_1(2h) = f'(x_\ast) + 2Kh + O(h^2).
\]

Thus, using \( P(h) = 2P_1(h) - P_1(2h) \) yields to:

\[
P(h) = f'(x_\ast) + O(h^2).
\]

Therefore, the method \( P \) is of order 2.

Exercise 2. 30pts

a) The vector fields \( f(t,y) = \cos y - \sin y \) is a smooth function (i.e. \( C^\infty \)), thus one can apply Cauchy-Lipschitz theorem and deduce that there exists a unique solution \( y(t) \) defined on a small interval \( t \in [0, \delta] \). Moreover, since \( f \) is bounded (\(|f(t,y)| \leq 2 \)), the solution exists for all time.

We use that \( f \) is bounded to deduce an upper and lower bound on \( y(t) \):

\[
-2 \leq y'(t) \leq 2 \quad \Rightarrow \quad -2t \leq y(t) - y(0) \leq 2t \quad \Rightarrow \quad -2t \leq y(t) \leq 2t.
\]

b) Let \( t_n = n\Delta t \). For \( 0 \leq t_n \leq 2 \), the Euler method \( \{y_n\}_n \) satisfies:

\[
|y(t_n) - y_n| \leq \frac{\Delta t \cdot M}{2L} (e^{Lt_n} - 1),
\]

with \( M = \max_{[0,2]} |y''(t)| \) and \( L = \max_D |\partial_y f(t,y)| \) with \( D = [0,2] \times \mathbb{R} \). Here:

\[
|y''(t)| = | - y'(t) \cdot \sin y(t) - y'(t) \cdot \cos y(t) | \leq |y'| \cdot |\sin y| + |y'| \cdot |\cos y| \leq 2 + 2 \leq 4.
\]

Thus, we take \( M = 4 \). Moreover:

\[
|\partial_y f(t,y)| = | - \sin y - \cos y | \leq 2.
\]

Therefore, we take \( L = 2 \). In order to have accuracy \( 10^{-2} \) at \( t = 2 \), a sufficient condition on \( \Delta t \) is given by:

\[
\Delta t \leq \frac{2L \cdot 10^{-3}}{M(e^{2L} - 1)} \approx 1.87 \cdot 10^{-5}.
\]
c) Solving \( f(y) = 0 \) yields: \( \cos y = \sin y \). There are 2 solutions on \((-\pi, \pi]\): \( y_* = \pm \frac{\pi}{4} \).

Thus, the equilibria are: \( y_* = \frac{\pi}{4} + k \cdot \pi \text{ for any integer } k \).

To study the stability of the equilibria, we estimate \( f'(y_*) = -\sin y_* - \cos y_* \).

Thus,

- if \( y_* = \frac{\pi}{4} + 2k\pi \), then \( f'(y_*) = -\sqrt{2} < 0 \) \( \Rightarrow \) a-stable.
- if \( y_* = -\frac{\pi}{4} + 2k\pi \), then \( f'(y_*) = \sqrt{2} > 0 \) \( \Rightarrow \) unstable.

Extra) Initially, \( y_0 \) is between the two equilibria: \( y_* = \pm \frac{\pi}{4} \). Thus, by uniqueness of solutions, \( y(t) \) has to stay between \(-\pi/4\) and \( \pi/4 \) (two solutions cannot cross). Moreover, \( y' = f(y) > 0 \text{ on } (-\pi/4, \pi/4) \), therefore the solution \( y(t) \) is increasing. Since \( y(t) \) is also bounded, \( y(t) \) converges: \( y(t) \xrightarrow{t \to \infty} y_\infty \).

Moreover, \( y_\infty \) is necessarily an equilibrium for \( f \), therefore \( y_\infty = \pi/4 \) since \( y(t) \) is increasing and bounded by \( \pi/4 \). Thus,

\( y(t) \xrightarrow{t \to \infty} \pi/4 \).

Exercise 3. [20pts]

a) Set \( \Delta t \) and \( y_0 \). The numerical scheme [Midpoint] estimates \((t_n, y_n)\) in the following way:

- \( t_n = n \cdot \Delta t \).
- \( y_n \) is estimated recursively using

\[
\begin{align*}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \Delta t/2, y_n + k_1 \Delta t/2) \\
y_{n+1} &= y_n + \Delta t \cdot k_2.
\end{align*}
\]

b) We suppose \( y_n = y(t_n) \). Then writing \( f = f(t_n, y_n) \):

\[
\begin{align*}
y_{n+1} &= y_n + \Delta t \cdot f(t_n + \Delta t/2, y_n + k_1 \Delta t/2) \\
&= y_n + \Delta t \cdot \left( f + \Delta t/2 \cdot \partial_t f + k_1 \Delta t/2 \cdot \partial_y f + O(\Delta t^2) \right) \\
&= y_n + \Delta t \cdot f + \frac{\Delta t^2}{2} \left( \partial_t f + f \cdot \partial_y f \right) + O(\Delta t^3).
\end{align*}
\]

Since \( y_n = y(t_n) \), we have \( f = y'(t_n) \) and \( \partial_t f + f \partial_y f = y''(t_n) \). Therefore:

\[
y_{n+1} = y(t_n) + \Delta t y'(t_n) + \frac{\Delta t^2}{2} y''(t_n) + O(\Delta t^3) = y(t_{n+1}) + O(\Delta t^3).
\]
Exercise 4. 25pts

a) Denoting \( y = x' \) yields to the following system:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= \frac{2x}{1+x^2}.
\end{align*}
\]

Integrating the equation \( x'' \cdot x' = \frac{2x}{1+x^2} \cdot x' \) leads to \( (x')^2/2 = \ln(1 + x^2) + C \). Thus, the quantity \( E(x, y) = y^2/2 - \ln(1 + x^2) \) remains constant along the solutions \((x(t), y(t))\).

b) The improved Euler method consists in the following steps: \((x_n, y_n)\) and \(\Delta t\) given

Auxiliary solution: \[
\begin{align*}
\tilde{x} &= x_n + \Delta t \cdot y_n \\
\tilde{y} &= y_n + \Delta t \cdot \frac{2x_n}{1+x_n^2}.
\end{align*}
\]

Update: \[
\begin{align*}
x_{n+1} &= x_n + \frac{\Delta t}{2} \cdot (y_n + \tilde{y}) \\
y_{n+1} &= y_n + \frac{\Delta t}{2} \cdot \left( \frac{2x_n}{1+x_n^2} + \frac{2\tilde{x}}{1+\tilde{x}^2} \right).
\end{align*}
\]

Extra) We investigate the stability of the equilibrium \((0, 0)\).

Let \( F(x, y) = \left( \frac{y}{\frac{2x}{1+x^2}} \right) \), we have:

\[
DF(0, 0) = \begin{bmatrix}
\frac{2}{(1+x^2)^2} & 1 \\
\frac{4x^2}{(1+x^2)^2} & 0
\end{bmatrix}_{(0,0)} = \begin{bmatrix}
0 & 1 \\
2 & 0
\end{bmatrix}.
\]

The eigenvalues of this matrix are the solutions of \( \lambda^2 - 2 = 0 \). Thus, \( \lambda = \pm \sqrt{2} \). Since one eigenvalue has a positive real part, the equilibrium \((0, 0)\) is unstable.