Ex 1. Let $\ell = 10$ and $\alpha = .5$ and consider the parabolic PDE with **periodic boundary conditions**:

\[
\begin{cases}
\partial_t u = \alpha^2 \partial_x^2 u , & 0 < x < \ell, \ t > 0 \\
u(x, 0) = \sin \frac{\pi}{\ell} x , & 0 < x < \ell \\
u(0, t) = u(\ell, t) , & t \geq 0.
\end{cases}
\]

a) Implement a finite-difference method to solve this PDE.

b) Consider the total mass of the solution:

\[m(t) = \int_0^\ell u(x, t) \, dx.\]

- Show (analytically) that the mass $m(t)$ is conserved.
  
  *Hint:* by periodicity $\partial_x u(0, t) = \partial_x u(\ell, t)$ for all $t \geq 0$.

- Plot the evolution of the mass $m(t)$ in time for the numerical solution.

c) Suppose we use as initial condition: $u_0(x) = 1 + \sin \frac{2\pi x}{\ell}$.

Can you guess what will be the stationary state? (i.e. find $u_*(x)$ such that $u(x, t) \xrightarrow{t \to \infty} u_*(x)$).

Confirm numerically your guess.

Ex 2. Let $\ell = 1$ and $c = 1$ and consider the wave equation:

\[
\begin{cases}
\partial_t^2 u = c^2 \partial_x^2 u , & 0 < x < \ell, \ t > 0 \\
u(x, 0) = \sin 2\pi x , & 0 < x < \ell \\
\partial_t u(x, 0) = 2\pi \sin 2\pi x , & 0 < x < \ell \\
u(x, 0) = u(x, \ell) , & t \geq 0.
\end{cases}
\]

a) Implement a finite-difference method to solve this PDE.

b) Test the scheme with:

- $\Delta x = 2 \cdot 10^{-1}, \Delta t = 10^{-1}$
- $\Delta x = 10^{-1}, \Delta t = 2 \cdot 10^{-1}.$

What do you observe?

c) Find the accuracy of the scheme using $\Delta t = .9 \cdot \Delta x$.

*Hint:* take $\Delta x = \frac{1}{10}, \frac{1}{30}, \frac{1}{70}, \frac{1}{200}, \frac{1}{550}, \frac{1}{1500}$ and compute the error with the exact solution $u(x, t) = \sin 2\pi x \cdot (\cos 2\pi t + \sin 2\pi t)$.