MAT 425: Homework 10 (04/13)

Ex 1. Let $\Omega = \{1 \leq x \leq 3,\ 0 \leq y \leq 1\}$. We consider the elliptic PDE:

$$\begin{cases} \partial_x^2 u + \partial_y^2 u = 0 & \text{on } \Omega \\ u(x,0) = \frac{1}{x},\ u(x,1) = \frac{x}{x^2+1} & \text{on } \partial\Omega \\ u(1,y) = \frac{1}{1+y^2},\ u(3,y) = \frac{3}{9+y^2} & \end{cases}$$

a) Show that $u_*(x,y) = \frac{x}{x^2+y^2}$ is the solution of the PDE.

b) Implement a Finite-Difference method to solve numerically the PDE.
Plot the numerical solution for $\Delta x = \Delta y = \frac{1}{10}$.

c) We want to study the numerical error of the solution using the norm L^∞:

$$\text{Error}(\Delta x, \Delta y) = \max_{i,j} |u_{i,j} - u_*(x_i, y_j)|,$$

with $x_i = 1 + i\Delta x,\ y_i = 0 + j\Delta y$.
Fix $\Delta y = 10^{-1}$ and compute $\text{Error}(\Delta x, \Delta y)$ for several Δx.
What do you observe?

Ex 2. Let $\ell = 10$ and consider the parabolic PDE:

$$\begin{cases} \partial_t u = \partial_x^2 u & \text{for } 0 < x < \ell,\ t > 0 \\ u(x,0) = \sin\frac{\pi x}{\ell} & \text{at } t = 0 \\ u(0,t) = u(\ell,t) = 0 & \text{for } t > 0. \end{cases}$$

a) Show that $u_*(x,t) = \exp\left(-\frac{t \cdot \pi^2}{\ell^2}\right) \cdot \sin\frac{\pi x}{\ell}$ is the solution of the PDE.

b) Implement a Finite-Difference method to solve numerically the PDE.
Plot the numerical solution at $t = 0, 1$ and 2 for $\Delta x = 2 \cdot 10^{-1}$ and $\Delta t = 10^{-2}$.

c) Fix $t = 1$. We want to study the numerical error using the norm L^∞:

$$\text{Error}(\Delta x, \Delta t) = \max_i |u_i^n - u_*(x_i, 1)| \quad \text{with } x_i = i\Delta x,\ n\Delta t = 1.$$

Fix $\Delta t = 2 \cdot 10^{-1}$ and compute $\text{Error}(\Delta x, \Delta t)$ for several Δx.
What do you observe? What is the optimal choice for Δx? Justify.