MAT 425: Homework 11 (04/20)

Ex 1. Let $\ell = 10$ and $\alpha = .5$ and consider the parabolic PDE with **periodic boundary conditions**:

$$\begin{cases} \partial_t u = \alpha^2 \partial_x^2 u & , 0 < x < \ell, \;\; t > 0 \\ u(x,0) = \sin \frac{\pi}{\ell} x & , 0 < x < \ell \\ \mathbf{u}(\mathbf{0},\mathbf{t}) = \mathbf{u}(\ell,\mathbf{t}) & , t \geq 0. \end{cases}$$

a) Implement a finite-difference method to solve this PDE.

b) Consider the total mass of the solution:

$$m(t) = \int_0^\ell u(x,t)\,dx.$$

- Show (analytically) that the mass $m(t)$ is conserved.
  Hint: by periodicity $\partial_x u(0,t) = \partial_x u(\ell,t)$ for all $t \geq 0$.
- Plot the evolution of the mass $m(t)$ in time for the numerical solution.

c) Suppose we use as initial condition: $u_0(x) = 1 + \sin \frac{2\pi x}{\ell}$.
Can you guess what will be the stationary state? (i.e. find $u_*(x)$ such that $u(x,t) \overset{t\to\infty}{\longrightarrow} u_*(x)$).
Confirm numerically your guess.

Ex 2. Let $\ell = 1$ and $c = 1$ and consider the wave equation:

$$\begin{cases} \partial_t^2 u = c^2 \partial_x^2 u & , 0 < x < \ell, \, t > 0 \\ u(x,0) = \sin 2\pi x & , 0 < x < \ell \\ \partial_t u(x,0) = 2\pi \sin 2\pi x & , 0 < x < \ell \\ \mathbf{u}(\mathbf{0},\mathbf{t}) = \mathbf{u}(\ell,\mathbf{t}) & , t \geq 0. \end{cases}$$

a) Implement a finite-difference method to solve this PDE.

b) Test the scheme with:

$$i)\; \Delta x = 2 \cdot 10^{-1}, \Delta t = 10^{-1} \quad , \quad ii)\; \Delta x = 10^{-1}, \Delta t = 2 \cdot 10^{-1}.$$

What do you observe?

c) Find the accuracy of the scheme using $\Delta t = .9 \cdot \Delta x$.
Hint: take $\Delta x = \frac{1}{10}, \frac{1}{30}, \frac{1}{70}, \frac{1}{200}, \frac{1}{550}, \frac{1}{1500}$ and compute the error with the exact solution $u(x,t) = \sin 2\pi x \cdot (\cos 2\pi t + \sin 2\pi t)$.