MAT 425: (Extra) Homework 12 (04/27)

Ex 1. Let $\ell = 20$ and $c = 2$ and consider the transport equation with periodic boundary conditions:

$$\begin{cases} \partial_t \rho + c\partial_x \rho = 0 & , 0 < x < \ell, \quad t > 0 \\ \rho(x,0) = \mathrm{e}^{-(x-10)^2} & , 0 < x < \ell \\ \rho(0,t) = \rho(\ell,t) & , t \geq 0. \end{cases}$$

a) Implement a finite-difference method to solve this PDE.
   Plot the solution at $t = 0$ and $t = 4$.

b) Let $\Delta t = .1$ and $\Delta x = .5$, study the numerical solution for large time t (i.e. $t = 10$, 50, 100...). What do you observe?

c) Take now $\Delta t = .2$ and $\Delta x = .5$ and study the numerical solution for large time t (i.e. $t = 10$, 50, 100...). What is the difference with b)?
   Can you find a couple $(\Delta t, \Delta x)$ such that the solution does not 'diffuse'?

Ex 2. We consider the transport equation

$$\begin{cases} \partial_t \rho + \partial_x \Big(x(1-x)\rho \Big) = 0 & , 0 < x < 2, \quad t > 0 \\ \rho(x,0) = \begin{cases} 1 & \text{if } .25 < x < 1.75 \\ 0 & \text{otherwise} \end{cases} & , t = 0 \\ \rho(0,t) = \rho(2,t) = 0 & , t \geq 0. \end{cases}$$

a) Write the ODE associated with this transport equation (i.e. $x' = ...$).
   What is the long time behavior of the solutions?

b) Propose a numerical scheme to solve the equation.
   Plot the solution for $\Delta x = .1$, $\Delta t = .05$ at $t = 4$.
   What is the limit as $t \to \infty$?