APM 576: Homework 1 (09/06)

1 Transport equation

Ex 1.

Write down an explicit formula for a function u solving the initial-value problem

$$\begin{array}{lcl} \partial_t u + b \cdot \nabla_x u + cu = 0 & , & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0(x) & , & x \in \mathbb{R}^n \end{array}$$

where $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constant and $u_0(x)$ is a given function.

Ex 2.

a) Find an explicit solution to:

$$\begin{array}{lcl} \partial_t u - x\partial_x u = 0 & , & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) & , & x \in \mathbb{R} \end{array}$$

where $u_0(x)$ is a given function.

b) Assuming u_0 is a continuous function, show that for any (fixed) $x \in \mathbb{R}$, $u(x,t)$ converges as $t \to -\infty$.

Remark: the solution $u(x,t)$ is transported by the flow $x' = -x$ but it does not preserve the mass: $\frac{d}{dt}\int_{x\in\mathbb{R}} u(x,t)\,dx \neq 0$. We *remedy* this in the following exercise.

Ex 3. [Divergence form]

a) Solve the transport equation:

$$\begin{array}{rlcl} \partial_t u + \partial_x(-xu) & = 0 & , & x \in \mathbb{R},\ t > 0 \\ u(x,0) & = u_0(x) & , & x \in \mathbb{R} \end{array}$$

where $u_0(x)$ is a given function.

b) Assuming that $u_0(x)$ is an integrable function (i.e. $u_0 \in L^1(\mathbb{R})$), show that the total mass is preserved: $\int_{x\in\mathbb{R}} u(x,t)\,dx = \int_{x\in\mathbb{R}} u_0(x)\,dx$.

2 Heat equation

Ex 4.

Assume $n = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$.

a) Show that $u_t = u_{xx}$ if and only if $v''(z) + z/2v'(z) = 0$. (*)
Show that the general solution to (*) is:

$$v(z) = c\int_0^z \mathrm{e}^{-c^2/4}\, ds + d.$$

b) Differentiate $u(x,t) = v(\frac{x}{\sqrt{t}})$ with respect to x and select the constant c properly, to obtain the fundamental solution Φ for $n = 1$. Explain why this procedure produces the fundamental solution.

Ex 5.

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu &= f \quad \text{in } \mathbb{R}^n \times (0,\infty) \\ u &= g \quad \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where c is a constant.

Ex 6. [Non-uniqueness]

Consider the Heat equation with zero as initial condition:

$$\begin{array}{lcl} \partial_t u = \partial_x^2 u & , & x \in \mathbb{R},\, t > 0 \\ u(x,0) = 0 & , & x \in \mathbb{R} \end{array}$$

a) Suppose the solution $u(x,t)$ can be written as a power series: $u(x,t) = \sum_{n=0}^{\infty} a_n(t)x^n$.
Show that the coefficients have to satisfy: $a_n'(t) = (n+2)(n+1)a_{n+2}$.

b) Deduce that a possible solution is of the form:

$$u(x,t) = \sum_{n=0}^{\infty} \frac{a_0^{(n)}(t)}{(2n)!} x^{2n}.$$

c) In order to satisfy the initial condition, the function $a_0(t)$ has to satisfy:

$$a_0^{(n)}(0) = 0,$$

where $.^{(n)}$ denotes the n^{th} derivative. Find a non-zero function satisfying these conditions.
Hint: consider a function of the form $\varphi(t) = \mathrm{e}^{-1/t^\alpha}$ *for* $t > 0$.

d) Conclude about the uniqueness of solution to the heat equation.

Remark. The function $a_0(t)$ has all its derivative equal to zero at $t = 0$ and yet the function $a_0(t)$ is not the zero function. We have actually defined a C^∞ function that is *not* analytic.

3 Laplace equation

Ex 7.

Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define $v(x) := u(Ox)$ for any $x \in \mathbb{R}^n$, then

$$\Delta v = 0.$$

Ex 8.

Give a direct proof that if $\mathbf{u} \in C^2(U) \cap C^0(\overline{U})$ is harmonic within a bounded open set U, then:

$$\max_{\overline{U}} \mathbf{u} = \max_{\partial U} \mathbf{u}.$$

(Hint: Define $\mathbf{u}_\varepsilon := \mathbf{u} + \varepsilon |x|^2$ for $\varepsilon > 0$, and show $\mathbf{u}_\varepsilon$ cannot attain its maximum over U at an interior point.)