MAT 475: Solution homework 1 (08/28)

1 Chapter 1

Ex 2. [4pts]

a) Let $f(x) = x^3 - 3x$. To find the equilibrium, we solve $f(x) = 0$:

$$x^3 - 3x = 0 \quad \Rightarrow \quad x(x^2-3) = 0 \quad \Rightarrow \quad x = 0, \pm\sqrt{3}.$$

To decide whether the equilibria are sinks, sources (or neither of them), we look at the sign of $f'(x) = 3x^2 - 3$.

- $x = 0$ is a sink ($f'(0) < 0$), 1 pt
- $x = \pm\sqrt{3}$ are source ($f'(\pm\sqrt{3}) > 0$).

b) Let $f(x) = x^4 - x^2$. Equilibria:

- $x = 0$ neither a source or a sink. 1 pt
- $x = 1$ is a source.
- $x = -1$ is a sink.

c) Let $f(x) = \cos x$. Equilibria:

- $x = \frac{\pi}{2} + 2k\pi$ are sinks. 1 pt
- $x = -\frac{\pi}{2} + 2k\pi$ are sources.

d) Let $f(x) = \sin^2 x$. Equilibria:

- $x = 0 + k\pi$ are neither sinks or sources. 1 pt

e) Let $f(x) = |1 - x^2|$. Equilibria:

- $x = \pm 1$ are neither sinks or sources.

Ex 7. [3pts]

a) The solution satisfying $x(0) = 4$ is given by: 1 pt

$$x(t) = \begin{cases} 4 & t < 5 \\ 2e^{5-t} + 2 & t \geq 5 \end{cases}$$

b) Starting at $x(0) = 3$

1 pt

$$x(t) = \begin{cases} -e^t + 4 & t < 5 \\ (c_0 - 2)e^{5-t} + 2 & t \geq 5 \end{cases}$$

with $c_0 = \lim_{t \to 5^-} x(t) = -e^5 + 4$.

c) All the solutions converge to 2: $x(t) \overset{t \to +\infty}{\longrightarrow} 2$. 1 pt

Ex 8.

Consider y_2 another solution to $x' = ax + f(t)$. We study the difference: $v(t) = y_2(t) - y(t)$ and find that

$$v' = ay_2 + f - (ay + f) = a(y_2 - y) = av$$

Therefore, $v' = ce^{at}$ where c is arbitrary. We deduce that $y_2 = y + ce^{at}$.

Ex 9.

a) Consider $x(t) = ce^{\int_0^t a(s)\,ds}$ where c is an arbitrary constant.

b) Suppose $x(0) \neq 0$ (otherwise the solution is simply $x(t) = 0$), we have:

$$\frac{x'}{x} = a(t) \;\Rightarrow\; \ln|x(t)| = \int a(s)\,ds + C \;\Rightarrow\; |x(t)| = e^{\int a(s)\,ds} \cdot e^C$$

Taking $c = \pm e^C$, we deduce the formula in a).

Ex 11. [3pts]

a) We solve the ODE using separation of variables and we suppose that $x(0) = x_0$:

$$\frac{x'}{x^2} = 1 \;\Rightarrow\; -\frac{1}{x(t)} + \frac{1}{x_0} = t \;\Rightarrow\; x(t) = \frac{1}{1/x_0 - t}.$$

1 pt

This computation is relevant only if $x_0 \neq 0$. If $x_0 = 0$, the solution is given by: $x(t) = 0$.

b) If $x_0 > 0$, the solution $x(t)$ is defined up to time $t = 1/x_0$. If $x_0 \leq 0$, then the solution is defined for all time. .5+.5 pt

c) We take a function that 'explodes' at $x = \pm 1$. For instance $x(t) = \tan(\frac{2t}{\pi})$. Then:

$$x'(t) = \frac{2}{\pi}\tan'(2t/\pi) = \frac{2}{\pi}\Big(1 + \tan^2(2t/\pi)\Big) = \frac{2}{\pi}(1 + x^2(t)).$$

1 pt

Thus, $x(t)$ is the solution of: $x' = \frac{2}{\pi}(1 + x^2)$ that satisfies $x(0) = 0$. The solution $x(t)$ is only defined on the interval $-1 < t < 1$.

Ex 12.

a) The constant function $x_1(t) = 0$ for all $t \geq 0$ is one solution to $x' = x^{1/3}$ with $x(0) = 0$. We can find other solutions using separation of variables:

$$\frac{x'}{x^{1/3}} = 1 \quad \Rightarrow \quad \frac{3}{2}x^{2/3} = t + C \quad \Rightarrow \quad x = \left(\frac{2}{3}(t+C)\right)^{3/2}.$$

That gives another solutions $x_2(t) = \left(\frac{2t}{3}\right)^{3/2}$. Now we can *combine* the solutions x_1 and x_2 to have as many solutions as we want. Take $t_1 > 0$ and consider the function:

$$x(t) = \begin{cases} 0 & 0 \leq t < t_1 \\ \left(\frac{2}{3}(t-t_1)\right)^{3/2} & t_1 \leq t \end{cases}$$

For any value of t_1, the function $x(t)$ is a solution to $x' = x^{1/3}$ with $x(0) = 0$ (see figure 2). Thus, we have infinitely many solutions.

b) Using separation of variables, we find:

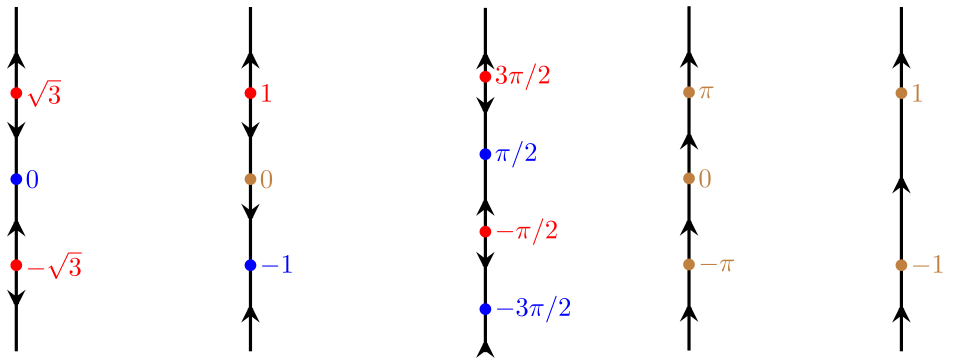
$$\frac{x'}{x} = \frac{1}{t} \quad \Rightarrow \quad \ln x = \ln t + C \quad \Rightarrow \quad x = Kt.$$

Thus, we need to have $x_0 = x(0) = 0$ otherwise there is no solution. Moreover, if $x(0) = 0$, then they are infinitely many solution (as K can be any value).

c) Using separation of variables, we find:

$$\frac{x'}{x} = \frac{1}{t^2} \quad \Rightarrow \quad \ln x = -\frac{1}{t} + C \quad \Rightarrow \quad x = K\mathrm{e}^{-1/t}.$$

Thus, at the limit $t \to 0$, we find: $x(0) = 0$. There exists infinitely many solutions as K can be any values.



a) $f(x) = x^3 - 3x$ **b)** $f(x) = x^4 - x^2$ **c)** $f(x) = \cos x$ **d)** $f(x) = \sin^2 x$ **e)** $f(x) = |1 - x^2|$

Figure 1: Phase line for **Ex 2**.

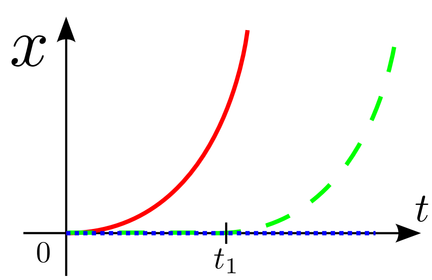


Figure 2: Three solutions of $x' = x^{1/3}$ with $x(0) = 0$: $x(t) = 0$ (blue), $x(t) = \left(\frac{2t}{3}\right)^{3/2}$ (red) and a 'mixture' (green).