APM 576: Homework 1 (09/06)

1 Transport equation

Ex 1.

The source term "$\partial_t \mathbf{u} = -c\mathbf{u}$" makes the function decays exponentially: $\mathbf{u}(x,t) = \mathrm{e}^{-ct}\mathbf{u}_0(x)$. To *remove* this effect, we consider the function: $\mathbf{v}(x,t) = \mathrm{e}^{ct}\mathbf{u}(x,t)$. It satisfies:

$$\partial_t \mathbf{v} = c\,\mathrm{e}^{ct}\mathbf{u} + \mathrm{e}^{ct}(-b\cdot\nabla_x \mathbf{u} - c\mathbf{u}) = -b\cdot\nabla_x(\mathrm{e}^{ct}\mathbf{u}) = -b\cdot\nabla_x \mathbf{v}.$$

Thus, $\partial_t \mathbf{v} + b\cdot\nabla_x \mathbf{v}$. We can solve exactly this equation since the characteristics are explicit ($\mathbf{x}' = b \Rightarrow \mathbf{x}(t) = \mathbf{x}_0 + bt$):

$$\mathbf{v}(x,t) = \mathbf{v}_0(x - bt) = \mathbf{u}_0(x - bt)$$

Therefore:

$$\mathbf{u}(x,t) = \mathrm{e}^{-ct}\mathbf{v}(x,t) = \mathrm{e}^{-ct}\mathbf{u}_0(x - bt).$$

Ex 2.

a) The characteristics are given by (see Fig. 1) :

$$\mathbf{x}' = -\mathbf{x} \Rightarrow \mathbf{x}(t) = \mathrm{e}^{-t}\mathbf{x}_0.$$

Thus, the solution is given by:

$$\mathbf{u}(x,t) = \mathbf{u}_0(\mathrm{e}^{+t}x).$$

b) If $\mathbf{u}_0$ is continuous, we can pass to the limit $t \to -\infty$ in the expression:

$$\lim_{t\to-\infty} \mathbf{u}(x,t) = \mathbf{u}_0(0).$$

The function becomes flat in this limit.

Ex 3. [Divergence form]

a) Along the characteristics $\mathbf{x}(t) = \mathrm{e}^{-t}\mathbf{x}_0$, the solution satisfies:

$$\begin{aligned} \frac{d}{dt}\mathbf{u}(\mathbf{x}(t),t) &= \mathbf{x}'\partial_x \mathbf{u} + \partial_t \mathbf{u} = -x\partial_x \mathbf{u} + \partial_t \mathbf{u} \\ &= \partial_x(-x\mathbf{u}) + \mathbf{u} + \partial_t \mathbf{u} = \mathbf{u}. \end{aligned}$$

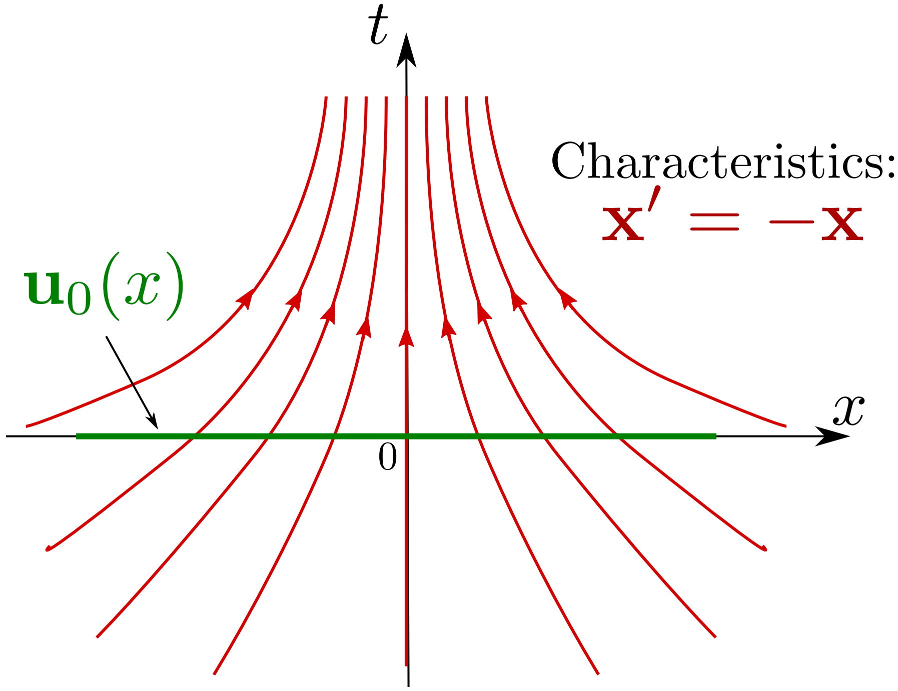

Figure 1: Characteristics for the transport equation: $\partial_t \mathbf{u} - x\partial_x \mathbf{u} = 0$.

Thus, denoting $\mathbf{z}(t) = \mathbf{u}(\mathbf{x}(t), t)$, we have: $\mathbf{z}' = \mathbf{z}$. Therefore,

$$\mathbf{u}(\mathbf{x}(t), t) = \mathbf{z}(t) = \mathbf{z}(0)\mathrm{e}^t = \mathbf{u}(\mathbf{x}(0), 0)\mathrm{e}^t = \mathbf{u}_0(\mathbf{x}(0))\mathrm{e}^t.$$

Thus,

$$\mathbf{u}(x, t) = \mathbf{u}_0(\mathrm{e}^t x)\mathrm{e}^t.$$

b) We estimate the total mass at time t **u**sing the change of coordinate $y = \mathrm{e}^t x$ (t is fixed)

$$\int_{x\in\mathbb{R}} \mathbf{u}(x, t)\, dx = \int_{x\in\mathbb{R}} \mathbf{u}_0(\mathrm{e}^t x)\mathrm{e}^t\, dx = \int_{y\in\mathbb{R}} \mathbf{u}_0(y)\, dy.$$

Thus, the total mass is preserved over time.

2 Heat equation

Ex 4.

We assume $\mathbf{u}(x, t) = \mathbf{v}(\frac{x}{\sqrt{t}})$.

a) We have:

$$\begin{aligned} \mathbf{u}_t &= -\frac{1}{2}\frac{x}{t^{3/2}}\mathbf{v}' = -\frac{1}{2}\frac{z}{t}\mathbf{v}' \\ \mathbf{u}_{xx} &= \partial_x\left(\frac{1}{\sqrt{t}}\mathbf{v}'\right) = \frac{1}{t}\mathbf{v}'' \end{aligned}$$

Thus, $\mathbf{u}_t = \mathbf{u}_{xx}$ iff $-\frac{1}{2}\frac{z}{t}\mathbf{v}'(z) = \frac{1}{t}\mathbf{v}''(z)$ and therefore

$$\mathbf{v}''(z) + z/2\mathbf{v}'(z) = 0.$$

We can solve this differential equation. Suppose $\mathbf{v} \neq 0$:

$$\frac{\mathbf{v}''}{\mathbf{v}'} = -\frac{z}{2} \quad \Rightarrow \quad \ln|\mathbf{v}'| = -z^2/4 + K \quad \Rightarrow \quad \mathbf{v}' = c\mathrm{e}^{-z^2/4}$$
$$\Rightarrow \quad \mathbf{v}(z) = c\int_{s=0}^{z} \mathrm{e}^{-s^2/4}\, ds \,+\, d.$$

b) We deduce:

$$\partial_x \mathbf{u} = \frac{1}{\sqrt{t}}\, c\,\mathrm{e}^{-x^2/4t}.$$

Thus, taking $c = 1/\sqrt{4\pi}$, we obtain: $\partial_x \mathbf{u} = \Phi_t$.

The function $\partial_x \mathbf{u}$ is necessarily the fundamental solution since:

i) it solves the heat equation: $\partial_t \mathbf{u} = \partial_x^2 \mathbf{u}$ implies $\partial_t(\partial_x \mathbf{u}) = \partial_x^2(\partial_x \mathbf{u})$.

ii) in the limit $t \to 0^+$, $\mathbf{u}(x,t)$ becomes a flat function with a *jump* at $x = 0$:

$$\lim_{t\to 0^+} \mathbf{u}(x,t) = \begin{cases} \lim_{y\to+\infty} \mathbf{v}(y) & \text{if } x > 0 \\ \lim_{y\to-\infty} \mathbf{v}(y) & \text{if } x < 0 \end{cases} = \begin{cases} 1/2 + d & \text{if } x > 0 \\ -1/2 + d & \text{if } x < 0 \end{cases}$$

since $\int_{\mathbb{R}} \mathrm{e}^{-s^2/4}\, ds = \sqrt{4\pi}$. Therefore, $\partial_x \mathbf{u}(x,0)$ is the Delta distribution.

Ex 5.

We use the same trick as in **Ex 2.**, letting $\mathbf{v}(x,t) = \mathrm{e}^{ct}\mathbf{u}(x,t)$. It satisfies:

$$\mathbf{v}_t(x,t) - \Delta \mathbf{v}(x,t) = \mathrm{e}^{ct} f(x,t),$$

and $\mathbf{v}(x,0) = g$. Thus, we find that:

$$\mathbf{v}(x,t) = \phi_t * g(x) + \int_0^t \phi_{t-s} * \mathrm{e}^{cs} f(.,s)(x)\, ds,$$

where ϕ_t is the fundamental solution and $'*'$ denotes the convolution operation. We conclude that $\mathbf{u}$ is given by:

$$\mathbf{u}(x,t) = \mathrm{e}^{-ct}\int_{y\in\mathbb{R}^n} \phi_t(x-y) g(y)\, dy + \int_0^t \int_{y\in\mathbb{R}^n} \phi_{t-s}(x-y)\mathrm{e}^{c(s-t)} f(y,s)\, ds.$$

Ex 6. [Non-uniqueness]

Consider the Heat equation with zero as initial condition:

$$\begin{array}{ll} \partial_t \mathbf{u} = \partial_x^2 \mathbf{u} \quad , & x \in \mathbb{R},\, t > 0 \\ \mathbf{u}(x,0) = 0 \quad , & x \in \mathbb{R} \end{array}$$

a) Assuming $\mathbf{u}(x,t) = \sum_{n=0}^{\infty} a_n(t)x^n$, we obtain:

$$\begin{aligned}\partial_t \mathbf{u} &= \sum_{n=0}^{\infty} a_n' x^n \\ \partial_{xx} \mathbf{u} &= \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)x^k.\end{aligned}$$

In order for the expression to be equal, we need to have: $a_{k+2}(k+2)(k+1) = a_k'$.

b) We deduce a recursive formula for the coefficients $a_{n+2} = \frac{a_n'}{(n+1)(n+2)}$ which leads to:

$$a_{2n} = \frac{a_0^{(n)}}{(2n)!}.$$

Assuming that $a_1 = 0$, we deduce the formula:

$$\mathbf{u}(x,t) = \sum_{n=0}^{\infty} \frac{a_0^{(n)}(t)}{(2n)!} x^{2n}.$$

c) We look for a function $\varphi(t) = a_0(t)$ that satisfy

$$\varphi^{(n)}(0) = 0,$$

for all n but that is different from the zero function. Taking

$$\varphi(t) = \begin{cases} \mathrm{e}^{-1/t^\alpha} & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

We find that:

$$\varphi'(t) = (\mathrm{e}^{-t^{-\alpha}})' = \alpha t^{-\alpha-1}\mathrm{e}^{-t^{-\alpha}}.$$

Notice that, for $\alpha > 0$, φ' is continuous at $t = 0$ since:

$$\lim_{t \longrightarrow 0^+} \varphi'(t) = \lim_{s \longrightarrow +\infty} \varphi'(1/s) = \lim_{s \longrightarrow +\infty} \alpha s^{\alpha+1}\mathrm{e}^{-s^\alpha} = 0.$$

Similarly, we have:

$$\varphi''(t) = P_2(1/t)\mathrm{e}^{-t^{-\alpha}} \qquad \text{with} \quad P_2(s) = \alpha(\alpha+1)s^{\alpha+2} + \alpha^2 s^{2(\alpha+1)}.$$

By the same argument, φ'' continuous at $t = 0$. More generally, we will have: $\varphi^{(n)}(t) = P_n(1/t)\mathrm{e}^{-s^\alpha}$ where P_n sum of monomial of order less or equal $n(a+1)$ (P_n polynomial if α integer) and $\varphi^{(n)}$ continuous at $t = 0$. Therefore, φ fits the condition $\varphi^{(n)}(0) = 0$.

d) We can now conclude and build a solution of the heat equation with zero initial condition that it is non zero:

$$\mathbf{u}(x,t) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(t)}{(2n)!} x^{2n}. \tag{1}$$

The solution to the heat equation is therefore not unique. But notice that for any $t > 0$, the solution (1) does not necessarily converge to zero as $|x| \to +\infty$.

Remark. To show rigorously that $u(x,t)$ is a solution, we need to prove that the sum in (1) is well-defined. It is a delicate question since we know very little on the behavior of $\varphi^{(n)}(t)$. However, one can show [see Fritz *Partial Differential Equations* p.172] that:

$$|\varphi^{(n)}(t)| \leq \frac{n!}{(\theta t)^n} \mathrm{e}^{-\frac{1}{2}t^{-\alpha}}$$

with $\theta \in (0,1)$. After several computations, one can deduce uniform summability of the sum and therefore write:

$$\lim_{t\longrightarrow 0^+} \mathbf{u}(x,t) = \lim_{t\longrightarrow 0^+} \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(t)}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \lim_{t\longrightarrow 0^+} \frac{\varphi^{(n)}(t)}{(2n)!} x^{2n} = 0.$$

3 Laplace equation

Ex 7.

Suppose $\mathbf{u}$ solution to Laplace's equation $\Delta \mathbf{u} = 0$ and O orthogonal matrix (i.e. $O^T = O^{-1}$). Consider $\mathbf{v}(x) = \mathbf{u}(Ox) = \mathbf{u}(y)$. We give two proofs of the results.

- **Using coordinates**: $y_i = \sum_j a_{ij} x_j$ with a_{ij} coefficients of O in the canonical basis. We obtain:

$$\frac{\partial \mathbf{v}}{\partial x_i} = \sum_j \frac{\partial \mathbf{u}}{\partial y_j} \frac{\partial y_j}{\partial x_i} = \sum_j \frac{\partial \mathbf{u}}{\partial y_j} a_{ji}.$$

  Thus,

$$\frac{\partial^2 \mathbf{v}}{\partial x_i^2} = \sum_j a_{ji} \left(\sum_k \frac{\partial^2 \mathbf{u}}{\partial y_j \partial y_k} a_{ki} \right) = \sum_{j,k} \frac{\partial^2 \mathbf{u}}{\partial y_j \partial y_k} a_{ji} a_{ki},$$

  and

$$\Delta \mathbf{v} = \sum_i \frac{\partial^2 \mathbf{v}}{\partial x_i^2} = \sum_{j,k} \frac{\partial^2 \mathbf{u}}{\partial y_j \partial y_k} \sum_i a_{ji} a_{ki}.$$

  However $\sum_i a_{ji} a_{ki} = (OO^T)_{jk}$ and therefore the sum is equal to zero if $j \neq k$ and 1 if $j = k$ (i.e. δ_{jk} Kronecker symbol). We conclude:

$$\Delta \mathbf{v}(x) = \sum_j \frac{\partial^2 \mathbf{u}}{\partial y_j \partial y_j} = \Delta \mathbf{u}(y) = 0.$$

- **Using linear perturbation**: we compute explicitly $\nabla \mathbf{v}$ and $D^2 \mathbf{v}$ (Hessian of $\mathbf{v}$)

$$\begin{aligned} \mathbf{v}(x+\varepsilon) &= \mathbf{u}(Ox + O\varepsilon) = \mathbf{u}(Ox) + \langle \nabla \mathbf{u}(Ox) \, , \, O\varepsilon \rangle + o(|\varepsilon|) \\ &= \mathbf{u}(y) + \langle O^T \nabla \mathbf{u}(y) \, , \, \varepsilon \rangle + o(|\varepsilon|). \end{aligned}$$

  Thus, $\nabla \mathbf{v}(x) = O^T \nabla \mathbf{u}(y)$. Similarly,

$$\begin{aligned} \nabla \mathbf{v}(x+\varepsilon) &= O^T \nabla \mathbf{u}(Ox + O\varepsilon) = O^T \nabla \mathbf{u}(Ox) + O^T D^2 \mathbf{u}(Ox) O\varepsilon + o(|\varepsilon|) \\ &= \nabla \mathbf{v}(x) + O^T D^2 \mathbf{u}(y) O\varepsilon + o(|\varepsilon|), \end{aligned}$$

and we deduce $D^2\mathbf{v}(x) = O^T D^2\mathbf{u}(y)O$. Therefore:

$$\begin{aligned}\Delta\mathbf{v}(x) &= \text{Trace}(D^2\mathbf{v}(x)) = \text{Trace}(O^T D^2\mathbf{u}(y)O) \\ &= \text{Trace}(OO^T D^2\mathbf{u}(y)) = \text{Trace}(D^2\mathbf{u}(y)) = \Delta\mathbf{u}(y) = 0,\end{aligned}$$

using $\text{Trace}(AB) = \text{Trace}(BA)$.

Ex 8.

[By contradiction] We suppose that there exists an interior point x_0 satisfying:

$$\mathbf{u}(x_0) > \max_{\partial U} \mathbf{u}.$$

Since the domain U is bounded, for $\varepsilon > 0$ small enough, we still have for $\mathbf{u}_\varepsilon(x) = \mathbf{u}(x) + \varepsilon|x|$

$$\mathbf{u}_\varepsilon(x_0) > \max_{x\in\partial U} \mathbf{u}_\varepsilon, \tag{2}$$

Notice that $\Delta\mathbf{u}_\varepsilon = 0 + n\varepsilon > 0$ (with n dimension of the problem). Consider now the maximum of $\mathbf{u}_\varepsilon$ on $\overline{U}$ and denote it y_0 (y_0 might be x_0). The point y_0 cannot be on the frontier of the domain (i.e. $y_0 \notin \partial U$) because of (2):

$$\mathbf{u}_\varepsilon(y_0) \geq \mathbf{u}_\varepsilon(x_0) > \max_{x\in\partial U} \mathbf{u}_\varepsilon.$$

Since $\mathbf{u}_\varepsilon$ is also regular, it must satisfy at its maximum:

$$\nabla\mathbf{u}_\varepsilon(y_0) = 0 \quad , \quad D^2\,\mathbf{u}_\varepsilon(y_0) \text{ negative definite.}$$

Contradiction: since $\Delta\mathbf{u}_\varepsilon > 0$, the Hessian cannot be negative definite at y_0.