APM 576: Homework 2 (09/27)

1 L^p and Hilbert spaces

Ex 1.

Consider the Banach space $L^\infty(\mathbb{R})$ with its usual norm $\|.\|_\infty$. Show that $C_c^0(\mathbb{R})$ (continuous function with compact support) are **not** dense in $L^\infty(\mathbb{R})$.

Remark. More generally, for any Ω open set of $\mathbb{R}^n$, $C_c^0(\Omega)$ is never dense in $L^\infty(\Omega)$.

Ex 2. [Riesz representation theorem]

Let H be a Hilbert space. We want to show that for any continuous linear form ℓ there exists a unique u_ℓ (i.e. the *representative* of ℓ) such that:

$$\ell(v) = \langle u_\ell, v\rangle \quad , \quad \text{for any } v \in H. \tag{1}$$

We denote by V the set (hyperplane) $V = \text{Ker}(\ell) = \{v \in H \,|\, \ell(v) = 0\}$.

a) If $V = H$, find u_ℓ.

b) If $V \neq H$, take $u_0 \notin V$ (i.e. $\ell(u_0) \neq 0$), denote p_0 its projection on V.
The vector $b = u_0 - p_0$ satisfies $b \perp V$, i.e. for $v \in V$, $\langle b, v\rangle = 0 = \ell(v)$.

Find a constant α such that $\langle \alpha\, b, u_0\rangle = \ell(u_0)$.
Conclude that $u_\ell = \alpha\, b$ satisfies (1).

c) Show that u_ℓ is unique.

2 Sobolev spaces

Ex 3.

Denote by Ω the open square $\{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0, \quad |x_2| < x_1 \\ 1 + x_1 & \text{if } x_1 < 0, \quad |x_2| < -x_1 \\ 1 - x_2 & \text{if } x_2 > 0, \quad |x_1| < x_2 \\ 1 + x_2 & \text{if } x_2 < 0, \quad |x_1| < -x_2 \end{cases}$$

For which $1 \leq p \leq \infty$ does u belong to $W^{1,p}(\Omega)$?

3 Approximation

Ex 4.

Let U, V open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta = 1$ on V and $\zeta = 0$ near ∂U. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify the indicator function $\mathbb{1}_W$.)

Ex 5.

Assume U is bounded and $U \subset\subset \cup_{i=1}^N V_i$. Show there exist C^∞ functions ζ_i ($i = 1 \ldots N$) such that:

$$\begin{cases} 0 \leq \zeta_i \leq 1, & \mathrm{Supp}(\zeta_i) \subset V_i \\ \sum_{i=1}^N \zeta_i = 1 & \text{on } U. \end{cases}$$

The function $\{\zeta_i\}_{i=1}^N$ form a partition of unity.

4 Trace

Ex 6.

Let Ω be bounded, with a C^1 boundary. Show that a "typical" function $u \in L^p(\Omega)$ ($1 \leq p < \infty$) does not have a trace on $\partial\Omega$. More precisely, prove there does not exist a **bounded** linear operator:

$$T \,:\, L^p(\Omega) \longrightarrow L^p(\partial\Omega)$$

such that $Tu = u|_{\partial\Omega}$ whenever $u \in C(\overline{\Omega}) \cap L^p(U)$.

5 Inequalities

Ex 7.

Integrate by parts to prove the interpolation inequality:

$$\|\nabla u\|_{L^2} \leq C\|u\|_{L^2}^{1/2}\|D^2 u\|_{L^2}^{1/2}$$

for all $u \in C_c^\infty(\Omega)$ where D^2u denotes the Hessian of u. Assume Ω is bounded, $\partial\Omega$ is smooth, and prove this inequality if $u \in N^2(\Omega) \cap H_0^1(\Omega)$.

: Hint: take sequences $\{v_k\}_k \subset C_c^\infty(\Omega)$ converging to u in $H_0^1(\Omega)$ and $\{w_k\}_k \subset C_c^\infty(\overline{\Omega})$ converging to u in $H^2(\Omega)$.

Ex 8.

Suppose Ω connected and $u \in W^{1,p}(\Omega)$ satisfies:

$$\nabla u = 0 \quad \text{a.e. in } \Omega.$$

Prove u is constant a.e. in U.