

# APM 576: Homework 2 (09/27)

## 1 $L^p$ and Hilbert spaces

**Ex 1.**

Consider the Banach space  $L^\infty(\mathbb{R})$  with its usual norm  $\|\cdot\|_\infty$ . Show that  $C_c^0(\mathbb{R})$  (continuous function with compact support) are **not** dense in  $L^\infty(\mathbb{R})$ .

**Remark.** More generally, for any  $\Omega$  open set of  $\mathbb{R}^n$ ,  $C_c^0(\Omega)$  is never dense in  $L^\infty(\Omega)$ .

**Ex 2.** [Riesz representation theorem]

Let  $H$  be a Hilbert space. We want to show that for any continuous linear form  $\ell$  there exists a unique  $u_\ell$  (i.e. the *representative* of  $\ell$ ) such that:

$$\ell(v) = \langle u_\ell, v \rangle \quad , \quad \text{for any } v \in H. \quad (1)$$

We denote by  $V$  the set (hyperplane)  $V = \text{Ker}(\ell) = \{v \in H \mid \ell(v) = 0\}$ .

a) If  $V = H$ , find  $u_\ell$ .

b) If  $V \neq H$ , take  $u_0 \notin V$  (i.e.  $\ell(u_0) \neq 0$ ), denote  $p_0$  its projection on  $V$ .  
The vector  $b = u_0 - p_0$  satisfies  $b \perp V$ , i.e. for  $v \in V$ ,  $\langle b, v \rangle = 0 = \ell(v)$ .

Find a constant  $\alpha$  such that  $\langle \alpha b, u_0 \rangle = \ell(u_0)$ .

Conclude that  $u_\ell = \alpha b$  satisfies (1).

c) Show that  $u_\ell$  is unique.

## 2 Sobolev spaces

**Ex 3.**

Denote by  $\Omega$  the open square  $\{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$ . Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0, \quad |x_2| < x_1 \\ 1 + x_1 & \text{if } x_1 < 0, \quad |x_2| < -x_1 \\ 1 - x_2 & \text{if } x_2 > 0, \quad |x_1| < x_2 \\ 1 + x_2 & \text{if } x_2 < 0, \quad |x_1| < -x_2 \end{cases}$$

For which  $1 \leq p \leq \infty$  does  $u$  belong to  $W^{1,p}(\Omega)$ ?

### 3 Approximation

**Ex 4.**

Let  $U, V$  open sets, with  $V \subset\subset U$ . Show there exists a smooth function  $\zeta$  such that  $\zeta = 1$  on  $V$  and  $\zeta = 0$  near  $\partial U$ . (Hint: Take  $V \subset\subset W \subset\subset U$  and mollify the indicator function  $\mathbb{1}_W$ .)

**Ex 5.**

Assume  $U$  is bounded and  $U \subset\subset \cup_{i=1}^N V_i$ . Show there exist  $C^\infty$  functions  $\zeta_i$  ( $i = 1 \dots N$ ) such that:

$$\begin{cases} 0 \leq \zeta_i \leq 1, & \text{Supp}(\zeta_i) \subset V_i \\ \sum_{i=1}^N \zeta_i = 1 & \text{on } U. \end{cases}$$

The functions  $\{\zeta_i\}_{i=1}^N$  form a partition of unity.

### 4 Trace

**Ex 6.**

Let  $\Omega$  be bounded, with a  $C^1$  boundary. Show that a “typical” function  $u \in L^p(\Omega)$  ( $1 \leq p < \infty$ ) does not have a trace on  $\partial\Omega$ . More precisely, prove there does not exist a **bounded** linear operator:

$$T : L^p(\Omega) \longrightarrow L^p(\partial\Omega)$$

such that  $Tu = u|_{\partial\Omega}$  whenever  $u \in C(\overline{\Omega}) \cap L^p(U)$ .

### 5 Inequalities

**Ex 7.**

Integrate by parts to prove the interpolation inequality:

$$\|\nabla u\|_{L^2} \leq C \|u\|_{L^2}^{1/2} \|D^2 u\|_{L^2}^{1/2}$$

for all  $u \in C_c^\infty(\Omega)$  where  $D^2 u$  denotes the Hessian of  $u$ . Assume  $\Omega$  is bounded,  $\partial\Omega$  is smooth, and prove this inequality if  $u \in C^2(\Omega) \cap H_0^1(\Omega)$ .

*Hint: take sequences  $\{v_k\}_k \subset C_c^\infty(\Omega)$  converging to  $u$  in  $H_0^1(\Omega)$  and  $\{w_k\}_k \subset C_c^\infty(\overline{\Omega})$  converging to  $u$  in  $H^2(\Omega)$ .*

**Ex 8.**

Suppose  $\Omega$  connected and  $u \in W^{1,p}(\Omega)$  satisfies:

$$\nabla u = 0 \quad \text{a.e. in } \Omega.$$

Prove  $u$  is constant a.e. in  $U$ .