MAT 475: Solution homework 3 (09/13)

1 Chapter 3

Ex 4. [4pts]

a) The characteristic polynomial is given by: $P(\lambda) = \lambda^2 + b\lambda + k$. Thus, the eigenvalues are:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4k}}{2}$$

The system has complex eigenvalues if $b^2 - 4k < 0$, repeated eigenvalues for $b^2 - 4k = 0$ and real and distinct eigenvalues for $b^2 - 4k > 0$. .5+.5+.5pt

b) **Complex eigenvalues:** $\underline{b^2 - 4k < 0}$. We look for the associated eigenvectors:

$$\lambda = \frac{-b + i\sqrt{|b^2 - 4k|}}{2} \quad , \qquad \begin{bmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

We can take: $\mathbf{w} = (1, \lambda) = \mathbf{u} + i\mathbf{v}$, with $\mathbf{u} = (1, -b/2)$, $\mathbf{v} = (0, \sqrt{|b^2 - 4k|}/2)$. Denote $\alpha = -b/2$ and $\beta = \frac{\sqrt{|b^2-4k|}}{2}$, we deduce that the general solution is given by:

$$\begin{aligned} \mathbf{x}(t) &= c_1 e^{\alpha t}(\cos\beta t\ \mathbf{u} - \sin\beta t\ \mathbf{v}) + c_2 e^{\alpha t}(\sin\beta t\ \mathbf{u} + \cos\beta t\ \mathbf{v}) \\ &= e^{\alpha t} \begin{pmatrix} c_1\cos\beta t + c_2 \sin\beta t \\ c_1\alpha\cos\beta t - c_1\beta\sin\beta t + c_2\alpha\sin\beta t + c_2\beta\cos\beta t \end{pmatrix}. \end{aligned}$$

.5pt

Repeated eigenvalues: $\underline{b^2 - 4k = 0}$. Eigenvector:

$$\lambda = -\frac{b}{2} \quad , \qquad \begin{bmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow \begin{bmatrix} b/2 & 1 \\ -b^2/4 & -b/2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

We can take: $\mathbf{u} = (1, -b/2)$. There is not a second linearly independent eigenvector. Thus, we solve the following equation:

$$(A - \lambda \mathrm{Id})\mathbf{v} = \mathbf{u} \quad \Rightarrow \quad \begin{bmatrix} b/2 & 1 \\ -b^2/4 & -b/2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -b/2 \end{pmatrix},$$

which gives $\mathbf{v} = (0, 1)$. Therefore, the general solution is given by:

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{u} \ + \ c_2 e^{\lambda t}(t\mathbf{u} + \mathbf{v}) \ = \ e^{\lambda t} \begin{pmatrix} c_1 + c_2 t \\ -\frac{b}{2}c_1 - \frac{bt}{2}c_2 + c_2 \end{pmatrix}$$

.5pt

Real and distinct eigenvalues: $\underline{b^2 - 4k > 0}$. Eigenvector:

$$\lambda_1 = \frac{-b - \sqrt{b^2 - 4k}}{2}, \mathbf{u}_1 = (1, \lambda_1) \quad , \quad \lambda_2 = \frac{-b + \sqrt{b^2 - 4k}}{2}, \mathbf{u}_2 = (1, \lambda_2).$$

The general solution is given by:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2.$$ .5pt

c) We take the initial condition $x_0 = (1, 0)$. We find the coefficients c_1, c_2 in each case.
Complex eigenvalues:

$$\mathbf{x}(0) = \mathbf{x}_0 \Rightarrow \begin{cases} c_1 = 1 \\ c_1\alpha + c_2\beta = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases}$$

Therefore, the motion of the mass is given by:

$$x(t) = e^{\alpha t}\cos\beta t,$$

with $\alpha = -b/2$ and $\beta = \sqrt{|b^2 - 4k|}/2$. The mass is oscillating around zero and the amplitude of the oscillating are decreasing since $\alpha < 0$. Thus, the mass will converge toward zero. .5pt

Repeated eigenvalues:

$$\mathbf{x}(0) = \mathbf{x}_0 \Rightarrow \begin{cases} c_1 = 1 \\ -\frac{b}{2}c_1 + c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = \frac{b}{2} \end{cases}$$

The motion of the mass is therefore:

$$x(t) = e^{-b/2t}(1 + bt/2).$$

The solution is going toward zero without any oscillation. .5pt

Real and distinct eigenvalues:

$$\mathbf{x}(0) = \mathbf{x}_0 \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ \lambda_1 c_1 + \lambda_2 c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_2 = 1 - c_1 \\ (\lambda_1 - \lambda_2)c_1 = -\lambda_2 \end{cases} \Rightarrow \begin{cases} c_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2} \\ c_1 = -\frac{\lambda_2}{\lambda_1 - \lambda_2} \end{cases}$$

The mass evolves according to: $x(t) = c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t}$. Since λ_1, $\lambda_2 < 0$, the solution converges exponentially fast to zero.

Ex 10. [3pts]

Since $\det(A) = 0$, the characteristic polynomial is given:

$$p_A(\lambda) = \lambda^2 - (a + d)\lambda = \lambda(\lambda - (a + d)).$$ .5pt

Therefore, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = a + d$. The associated eigenvectors are: .5pt

$$\lambda_1 = 0, \quad \mathbf{v}_1 = (d, -c) \quad , \quad \lambda_2 = a + d, \quad \mathbf{v}_2 = (a, c).$$ .5pt

Thus, the general solution is given by:

$$\mathbf{x}(t) = c_1 \begin{pmatrix} d \\ -c \end{pmatrix} + c_2 \mathrm{e}^{(a+d)t} \begin{pmatrix} a \\ c \end{pmatrix}.$$ .5pt

We sketch the phase portrait in the case $a + d > 0$ in figure 1. 1pt

Ex 11.

We solve the linear system:

$$\begin{cases} x' = y \\ y' = 0 \end{cases} \Rightarrow \begin{cases} x(t) = x_0 + y_0 t \\ y(t) = y_0 \end{cases}$$

The phase portrait is given in figure 2.

2 Chapter 4

Ex 2. [3pts]

Here, $\det(A) = a^2 - b^2$ and $\mathrm{trace}(A) = 2a$. Thus, the saddle points appear when $|b| > |a|$. Since $D \leq T^2/4$ (i.e. $a^2 - b^2 \leq a^2$), there is no spiral solution, only nodes which are either stable ($a < 0$) or unstable ($a > 0$). 1pt 1pt

The phase diagram is given in figure 3. 1pt

Ex 3.

Writing the equation as a linear system leads to

$$\mathbf{x}' = A\mathbf{x} \quad \text{, with } A = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}.$$

Thus, $\det(A) = k$ and $\mathrm{trace}(A) = -b$. Therefore, the phase diagram in the $b - k$ plane is the same as the usual phase diagram in $T - D$ (trace-determinant) phase, except that sinks and sources are reverse (see figure 4).

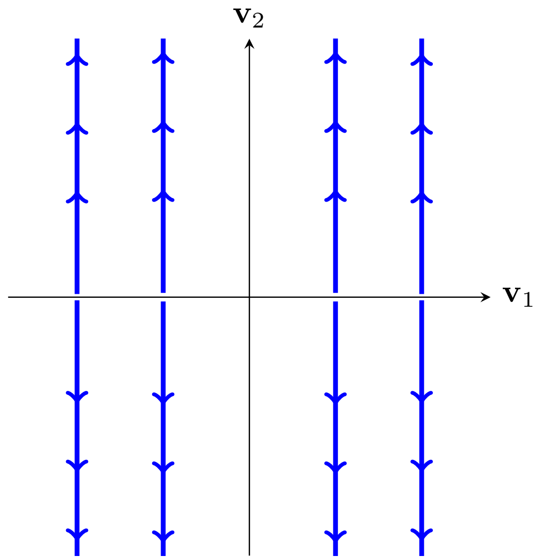

Figure 1: Phase portrait for **Ex 10**.

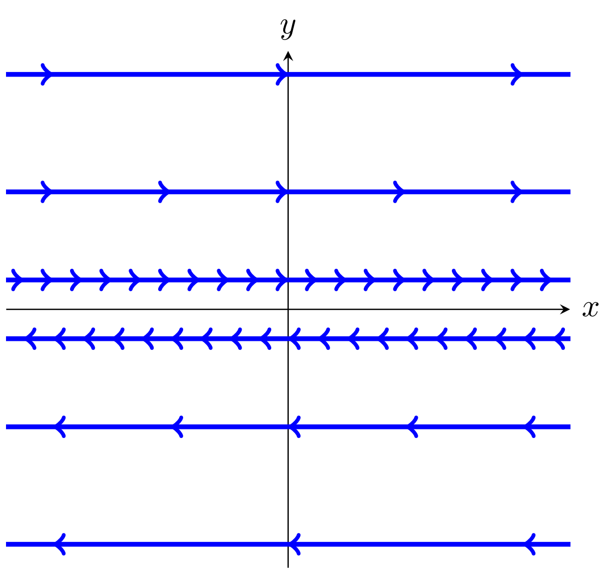

Figure 2: Phase portrait for **Ex 11**.

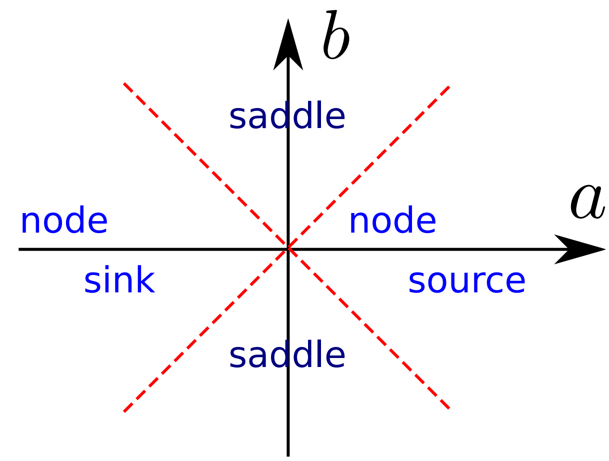

Figure 3: Phase diagram for **Ex. 2**.

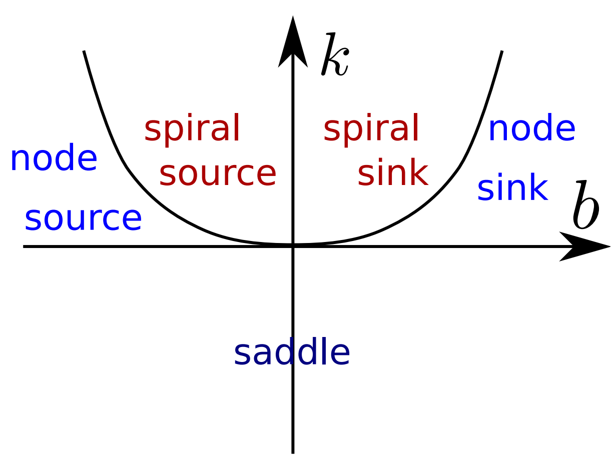

Figure 4: Phase diagram for **Ex. 3**.