MAT 475: Solution homework 2 (09/07)

1 Chapter 2

Ex 2. [3pts]

a) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$. To find the eigenvalues/eigenvectors of A, we compute its characteristic polynomial:

$$p_A(\lambda) = \det(A - \lambda Id) = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

Thus, $\lambda_1 = 1$ and $\lambda_2 = 3$ are the two eigenvalues of A. To find the eigenvector associated with each eigenvalue, we look for a vector $\mathbf{x}$ such that:

$$\lambda_1 = 1 \quad : \qquad (A - \text{Id})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can take $\mathbf{u}_1 = (1, 0)$. We proceed similarly for λ_2:

$$\lambda_2 = 3 \quad : \qquad (A - 3\text{Id})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We take as a solution the vector $\mathbf{u}_2 = (1, 1)$.

As a consequence, the general solution to $\mathbf{x}' = A\mathbf{x}$ is given by:

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2 = \begin{pmatrix} c_1 e^{t} & + & c_2 e^{3t} \\ & & c_2 e^{3t} \end{pmatrix}$$ 1pt

b) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. The eigenvalues/eigenvectors of A are given by:

$$\lambda_1 = 0,\ \mathbf{u}_1 = (2, -1) \quad , \quad \lambda_2 = 7,\ \mathbf{u}_2 = (1, 3).$$

Thus, the general solution is:

$$\mathbf{x}(t) = \begin{pmatrix} 2c_1 & + & c_2 e^{7t} \\ -c_1 & + & 3c_2 e^{7t} \end{pmatrix}$$ 1pt

c) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. The eigenvalues/eigenvectors of A are given by:

$$\lambda_1 = -1,\ \mathbf{u}_1 = (1, -1) \quad , \quad \lambda_2 = 2,\ \mathbf{u}_2 = (2, 1).$$

Thus, the general solution is:

$$\mathbf{x}(t) = \begin{pmatrix} c_1 \mathrm{e}^{-t} & + & 2c_2 \mathrm{e}^{2t} \\ -c_1 \mathrm{e}^{-t} & + & c_2 \mathrm{e}^{2t} \end{pmatrix}$$

d) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -3 \end{bmatrix}$. The eigenvalues/eigenvectors of A are given by:

$$\begin{aligned} \lambda_1 &= -1 - \sqrt{10},\ \ \mathbf{u}_1 = (-2, 1-\lambda_1) = (-2, 2 + \sqrt{10}) \\ \lambda_2 &= -1 + \sqrt{10},\ \ \mathbf{u}_2 = (-2, 1-\lambda_2) = (-2, 2 - \sqrt{10}) \end{aligned}$$

Thus, the general solution is:

$$\mathbf{x}(t) = \begin{pmatrix} -2c_1 \mathrm{e}^{(-1-\sqrt{10})t} & + & -2c_2 \mathrm{e}^{(-1+\sqrt{10})t} \\ (2+\sqrt{10})c_1 \mathrm{e}^{(-1-\sqrt{10})t} & + & (2-\sqrt{10})c_2 \mathrm{e}^{(-1+\sqrt{10})t} \end{pmatrix}.$$ 1pt

Ex 3.

To determine which direction fields correspond to which matrix A, one way is to 'look for' the eigenvectors, i.e. the one that point toward the origin (if $\lambda < 0$) or leave from the origin ($\lambda > 0$). From there we deduce the matching:

$$(1, c) \quad (2, b) \quad (3, d) \quad (4, a).$$

Ex 6. [4pts]

Introducing $y = x'$, we deduce that the linear differential equation associated with the harmonic oscillators is given by:

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \ \ A = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}.$$ .5pt

The characteristic polynomial is given by:

$$p_A(\lambda) = \lambda^2 + b\lambda + k.$$

Thus, the eigenvalues are:

$$\lambda_{1,2} = \frac{-b + \pm\sqrt{b^2 - 4k}}{2}.$$ .5pt

The eigenvalues are real and distinct if and only if: $b^2 - 4k > 0$. In this cases, the eigenvectors are given by:

$$\begin{bmatrix} -\lambda & 1 \\ -k & -b - \lambda \end{bmatrix} \mathbf{u} = 0$$

To find a solution to this system, we only need to 'cancel' the first equation. The second equation will be automatically satisfy as the equations are linearly dependent. Thus, we can take:

$$\lambda_1 = \frac{-b-\sqrt{b^2-4k}}{2} \quad , \quad \mathbf{u}_1 = (1,\lambda_1)$$
$$\lambda_2 = \frac{-b+\sqrt{b^2-4k}}{2} \quad , \quad \mathbf{u}_2 = (1,\lambda_2)$$

1pt

Therefore, the general solution is:

$$\mathbf{x}(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} & + & c_2 e^{\lambda_2 t} \\ \lambda_1 c_1 e^{\lambda_1 t} & + & \lambda_2 c_2 e^{\lambda_2 t} \end{pmatrix}.$$

.5pt

To satisfy the initial condition, we need to have:

$$\mathbf{x}(0) = (0,1) \quad \Rightarrow \quad \begin{cases} c_1 + c_2 = 0 \\ \lambda_1 c_1 + \lambda_2 c_2 = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} c_2 = -c_1 \\ c_1(\lambda_1 - \lambda_2) = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} c_2 = \frac{1}{\sqrt{b^2-4k}} \\ c_1 = -\frac{1}{\sqrt{b^2-4k}} \end{cases}$$

.5pt

Thus, the solution is given by:

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = \frac{1}{\sqrt{b^2-4k}}(e^{\lambda_1 t} - e^{\lambda_2 t}).$$

Both eigenvalues λ_1 and λ_2 are negative since $b > \sqrt{b^2-4k}$, the solution will converge to zero. Moreover, for $t > 0$, we have $x(t) > 0$. Therefore, the solution does not oscillate, it goes 'directly' toward zero. 1pt

Interpretation. When $b^2 > 4k$, the 'friction' coefficient b is so large compare to the oscillator force k that there is not even one oscillation.

Ex 10.

We would like to have as a solution:

$$\begin{cases} x(t) = t \\ y(t) = 1 \end{cases} \quad \Rightarrow \quad \begin{cases} x' = 1 \\ y' = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x' = y \\ y' = 0 \end{cases}$$

Thus, for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the solution $(t,1)$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

We sketch the direction fields of this equation in figure 1. We can solve explicitly the differential equations:

$$\begin{cases} x' & = & y \\ y' & = & 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x' & = & c_1 \\ y(t) & = & c_1 \end{cases} \quad \Rightarrow \quad \begin{cases} x(t) & = & c_1 t + c_2 \\ y(t) & = & c_1 \end{cases}$$

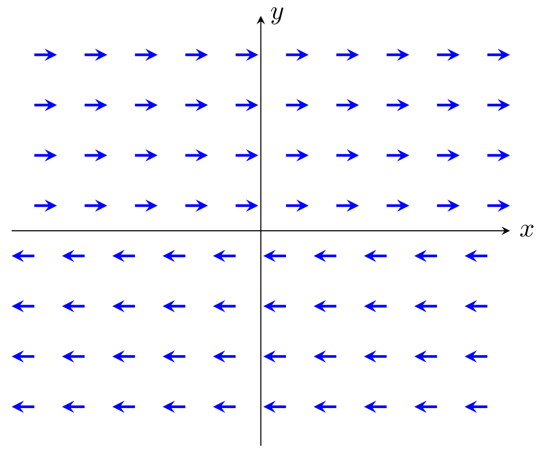

Figure 1: Direction fields of **Ex 10**: $x' = y$, $y' = 0$.

2 Chapter 3

Ex 2.

i) Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues/eigenvectors of A are given by:

$$\lambda_1 = -1,\ \mathbf{u}_1 = (1, -1) \quad , \quad \lambda_2 = 1,\ \mathbf{u}_2 = (1, 1).$$

Thus, the matrix of change of coordinates is defined as:

$$T = [\mathbf{u}_1\, \mathbf{u}_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and satisfies:

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The solution to $\mathbf{y}' = D\mathbf{y}$ with $D = T^{-1}AT$ is given by:

$$\mathbf{y}(t) = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{t} \end{pmatrix}.$$

Thus, the general solution to $\mathbf{x}' = A\mathbf{x}$ is:

$$\mathbf{x}(t) = T\mathbf{y}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 e^{t} \\ -c_1 e^{-t} + c_2 e^{t} \end{pmatrix}.$$

The phase portrait are given in figure 2.

iii) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. The eigenvalues/eigenvectors of A are given by:

$$\lambda = \frac{1 \pm i\sqrt{3}}{2}, \ \mathbf{u} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \pm i \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}.$$

Let $T = \begin{bmatrix} 1 & \sqrt{3} \\ -2 & 0 \end{bmatrix}$. We have:

$$T^{-1}AT = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \qquad \text{with } \alpha = \frac{1}{2}, \ \beta = \frac{\sqrt{3}}{2}.$$

The solution to $\mathbf{y}' = R\mathbf{y}$ with $R = T^{-1}AT$ is given by:

$$\mathbf{y}(t) = \mathrm{e}^{t/2} \begin{pmatrix} c_1 \cos \beta t + c_2 \sin \beta t \\ -c_1 \sin \beta t + c_2 \cos \beta t \end{pmatrix}.$$

Thus, the general solution to $\mathbf{x}' = A\mathbf{x}$ is:

$$\mathbf{x}(t) = T\mathbf{y}(t) = \mathrm{e}^{t/2} \begin{pmatrix} (c_1 + \sqrt{3}c_2) \cos(\beta t) + (c_2 - \sqrt{3}c_1) \sin(\beta t) \\ -2c_1 \sin \beta t + 2c_2 \cos \beta t \end{pmatrix}.$$

The phase portrait are given in figure 2.

v) Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$. The eigenvalues/eigenvectors of A are given by:

$$\lambda_1 = -1 - \sqrt{3}, \ \mathbf{u}_1 = (1, -2 - \sqrt{3}) \quad , \quad \lambda_2 = -1 + \sqrt{3}, \ \mathbf{u}_2 = (1, -2 + \sqrt{3}).$$

Let $T = \begin{bmatrix} 1 & 1 \\ -2 - \sqrt{3} & -2 + \sqrt{3} \end{bmatrix}$ such that

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

The solution to $\mathbf{y}' = D\mathbf{y}$ with $D = T^{-1}AT$ is given by:

$$\mathbf{y}(t) = \begin{pmatrix} c_1 \mathrm{e}^{\lambda_1 t} \\ c_2 \mathrm{e}^{\lambda_2 t} \end{pmatrix}.$$

Thus, the general solution to $\mathbf{x}' = A\mathbf{x}$ is:

$$\mathbf{x}(t) = T\mathbf{y}(t) = \begin{pmatrix} c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t} \\ c_1(-2 - \sqrt{3})\mathrm{e}^{\lambda_1 t} + c_2(-2 + \sqrt{3})\mathrm{e}^{\lambda_2 t} \end{pmatrix}.$$

The phase portrait are given in figure 2.

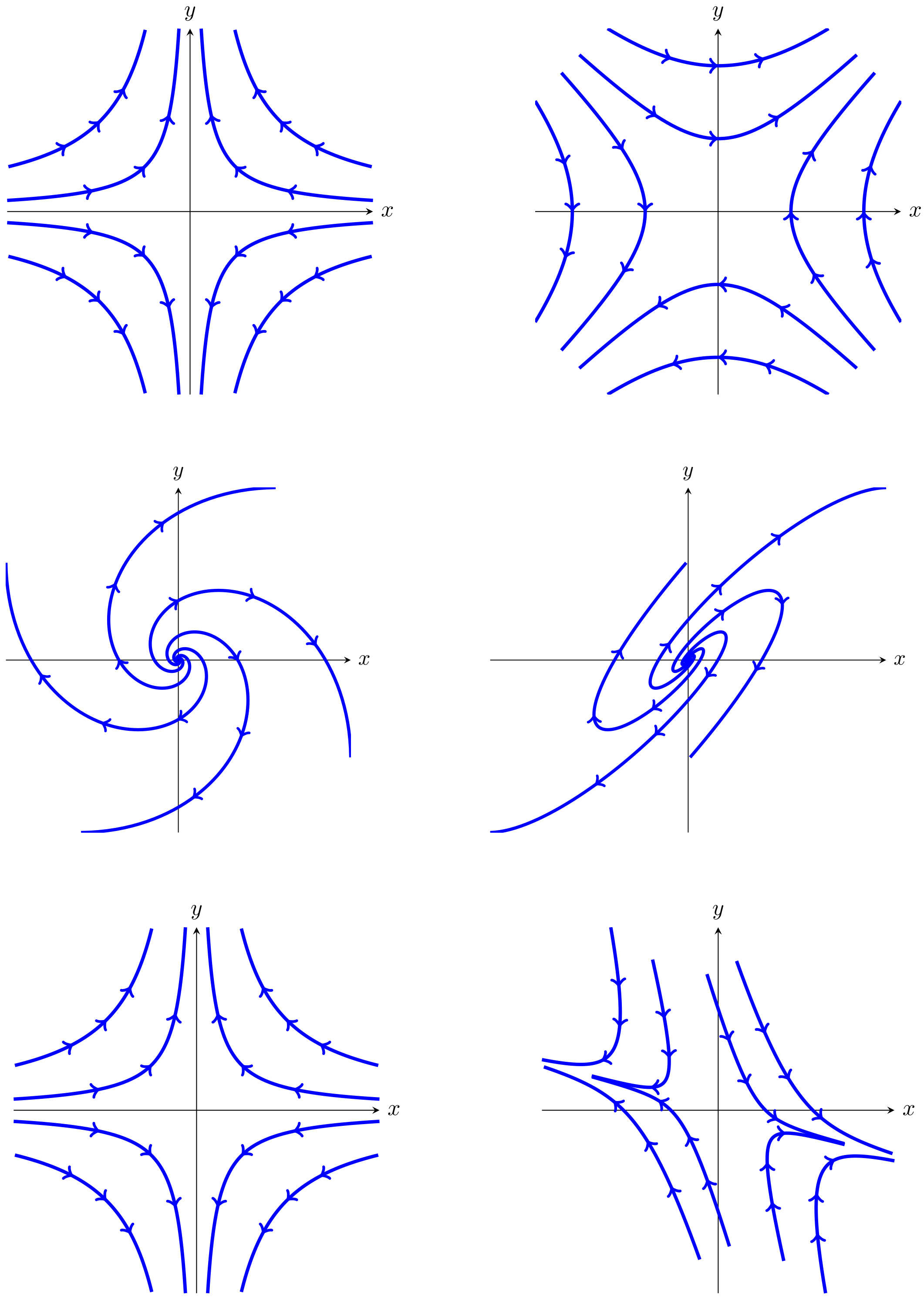

Figure 2: Phase portrait of the the 3 differential equations: top (i), middle (iii) and bottom (v).

Ex 3. [3pts]

a) Let $x'' + x' + x = 0$. The characteristic equation is given by:

$$\lambda^2 + \lambda + 1 = 0.$$

The roots are: $\lambda_{1,2} = \frac{-1 \pm i\sqrt{3}}{2} = \alpha + i\beta$ with $\alpha = -1/2$ and $\beta = \sqrt{3}/2$. Thus, the general solution is: .5pt

$$x(t) = \mathrm{e}^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$ .5pt

b) Let $x'' + 2x' + x = 0$. The characteristic equation is given by:

$$\lambda^2 + 2\lambda + 1 = 0.$$

There is a repeated eigenvalue: $\lambda = -1$. We need to investigate the linear system: .5pt

$$\mathbf{x}' = A\mathbf{x} \qquad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

The associated eigenvectors with $\lambda = -1$ are:

$$(A + \mathrm{Id})\mathbf{x} = 0 \Rightarrow \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There is only one eigenvector $\mathbf{u}_1 = (1, -1)$ which gives the solution: .5pt

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-t} \mathbf{u}_1,$$

thus $x(t) = c_1 \mathrm{e}^{-t}$. To find the other solution, we use the *undetermined coefficients* method which gives as a second solution .5 pt

$$y(t) = c_2 t \mathrm{e}^{-t}.$$

Indeed,

$$y'' + 2y' + y = c_2\Big(t\mathrm{e}^{-t} - 2\mathrm{e}^{-t} \; + \; 2\mathrm{e}^{-t} - 2t\mathrm{e}^{-t} \; + \; t\mathrm{e}^{-t}\Big) = 0.$$

Therefore, the general solution is given by:

$$x(t) = c_1 \mathrm{e}^{-t} + c_2 t \mathrm{e}^{-t}.$$ .5 pt