MAT 475: Solution homework 4 (09/20)

1 Chapter 5

Ex 2. [3pts]

a) The characteristic polynomial is given by:

$$p_A(\lambda) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda(1-\lambda)(-\lambda) - 1(1-\lambda) = (\lambda^2-1)(1-\lambda) = -(1-\lambda)^2(1+\lambda).$$

Thus, the eigenvalues are: $\lambda_1 = -1$, $\lambda_2 = 1$.
Eigenvectors:

$$\lambda_1 = -1 : \;\; \mathbf{v}_1 = (1,0,-1), \quad \lambda_3 = 1 : \;\; \mathbf{v}_2 = (1,0,1),\, \mathbf{w}_2 = (0,1,0).$$ .5pt

b) $p_A(\lambda) = (\lambda^2 - 3)(2-\lambda)$. Eigenvalues: $\lambda_1 = -\sqrt{3}$, $\lambda_2 = \sqrt{3}$, $\lambda_3 = 2$.
Eigenvectors:

.5pt
$$\lambda_1 = -\sqrt{3} : \;\; \mathbf{v}_1 = (1,0,-\sqrt{3}), \quad \lambda_2 = \sqrt{3} : \;\; \mathbf{v}_2 = (1,0,\sqrt{3}), \quad \lambda_3 = 2 : \;\; \mathbf{v}_3 = (0,1,0).$$

c) Eigenvalues/eigenvectors:

$$\lambda_1 = 0 : \;\; \mathbf{v}_1 = (1,-1,0),\, \mathbf{w}_1 = (0,1,-1), \quad \lambda_2 = 3 : \;\; \mathbf{v}_2 = (1,1,1).$$ .5pt

d) Eigenvalues/eigenvectors:

$$\lambda_{1,2} = \pm 2i : \;\; \mathbf{w} = (\pm i, 0, -1), \quad \lambda_3 = 2 : \;\; \mathbf{v}_3 = (0,1,0).$$ .5pt

e) The characteristic polynomial is given by:

$$\begin{aligned} p_A(\lambda) &= \begin{vmatrix} 3-\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 2 & 2 \\ 1 & -2 & -1-\lambda & -4 \\ -1 & 0 & 0 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda)\begin{vmatrix} 1-\lambda & 2 & 2 \\ -2 & -1-\lambda & -4 \\ 0 & 0 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1-\lambda & 2 \\ 1 & -2 & -1-\lambda \\ -1 & 0 & 0 \end{vmatrix} \\ &= (3-\lambda)^2\big((1-\lambda)(-1-\lambda)+4\big) \; + \; 1\big((1-\lambda)(-1-\lambda)+4\big) \\ &= \big((3-\lambda)^2+1\big)(\lambda^2+3). \end{aligned}$$ .5pt

Thus, the eigenvalues are: $\lambda_{1,2} = 2 \pm i$, $\lambda_{3,4} = \pm i\sqrt{3}$.
Eigenvectors:

$$\lambda_{1,2} = 3 \pm i \ : \ \ \mathbf{w}_{1,2} = (1, 0, \mp i, \pm i), \quad \lambda_{3,4} = \pm i\sqrt{3} \ : \ \ \mathbf{w}_{3,4} = (0, -1 \mp i\sqrt{3}, 2, 0).$$ .5pt

Ex 3.

The characteristic polynomial is given by:

$$p_A(\lambda) = (\lambda^2 - ca)(b - \lambda).$$

Thus, $\lambda = b$ is always an eigenvalue.

- Real eigenvalues: $ca \geq 0$.
- Complex eigenvalues: $ca < 0$.
- Repeated eigenvalues: $ca = 0$ or $ca = b$.

Ex 5. [3pts]

a) $p_A(\lambda) = -(\lambda - 1)^2(\lambda + 1)$. There is actually two eigenvectors associated to $\lambda = 1$: $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (1, 0, 1)$. Taking as a third vector $\mathbf{v}_3 = (1, 0, -1)$, we find that:

$$T^{-1}AT = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$ .5pt

with $T = [\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3]$.

b) Here, $p_A(\lambda) = (1 - \lambda)^3$. But there are only 2 eigenvectors. Thus, the matrix can be put in the form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. .5pt

c) $p_A(\lambda) = (1 - \lambda)(\lambda^2 + 1)$. Thus, one real eigenvalue ($\lambda_1 = 1$) and two conjugate complex eigenvalues ($\lambda_{2,3} = \pm i$). The matrix can be put in the form: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. .5pt

d) $p_A(\lambda) = -(1 - \lambda)^2(\lambda + 1)$. Only one eigenvector associated to $\lambda = 1$, thus the canonical form is: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. .5pt

e) $p_A(\lambda) = (1 - \lambda)\lambda(\lambda - 2)$. There are three eigenvectors, thus the canonical form is: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. .5pt

f) $p_A(\lambda) = (1-\lambda)(\lambda^2 - 2\lambda - 1)$. Three real and distinct eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 1-\sqrt{2}$, $\lambda_3 = 1+\sqrt{2}$, thus the canonical form is: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\sqrt{2} & 0 \\ 0 & 0 & 1+\sqrt{2} \end{bmatrix}$. .5pt

Ex 11.

We give two proofs:

- *with the determinant*. Suppose A or B are non-invertible then $\det(A) = 0$ or $\det(B) = 0$. Thus

$$\det(AB) = \det(A)\det(B) = 0.$$

Therefore, AB is non-invertible.

- *without the determinant*. Suppose first that B is non-invertible. Then there exists a vector $\mathbf{v} \neq \mathbf{0}$ such that: $B\mathbf{v} = \mathbf{0}$. Thus,

$$AB\mathbf{v} = A\mathbf{0} = \mathbf{0}.$$

Therefore, AB is not invertible since $AB\mathbf{v} = \mathbf{0}$ with $\mathbf{v} \neq \mathbf{0}$.
Suppose now B is invertible but A is not. Then, there exists $\mathbf{u} \neq \mathbf{0}$ such that $A\mathbf{u} = \mathbf{0}$. Take $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{u}$ ($\mathbf{u}$ exists as B is invertible). Then,

$$AB\mathbf{v} = A\mathbf{u} = \mathbf{0}.$$

Thus, AB is not invertible.

2 Chapter 6

Ex 1. [4pts]

a) $p_A(\lambda) = -(1-\lambda)^2(\lambda+1)$. Eigenvalues/eigenvectors:

$$\lambda = -1: \ \mathbf{v}_1 = (1,0,-1) \quad , \quad \lambda = 1: \ \mathbf{v}_2 = (1,0,1), \ \mathbf{v}_3 = (0,1,0).$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-t}\mathbf{v}_1 + c_2 \mathrm{e}^{t}\mathbf{v}_2 + c_3 \mathrm{e}^{t}\mathbf{v}_3.$$

.5pt

b) Eigenvalues/eigenvectors:

$$\lambda = 0: \ \mathbf{v}_1 = (1,0,-1) \quad , \quad \lambda = 1: \ \mathbf{v}_2 = (0,1,0) \quad , \quad \lambda = 2: \ \mathbf{v}_3 = (1,0,1).$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 + c_2 \mathrm{e}^{t}\mathbf{v}_2 + c_3 \mathrm{e}^{2t}\mathbf{v}_3.$$

.5pt

c) Eigenvalues/eigenvectors:

$$\lambda = 1: \ \mathbf{v}_1 = (0,0,1) \quad , \quad \lambda = i: \ \mathbf{w} = (1,i,i) = \mathbf{u} + i\mathbf{v},$$

with $\mathbf{u} = (1,0,0)$ and $\mathbf{v} = (0,1,1)$. General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{t}\mathbf{v}_1 + c_2\Big(\cos t\,\mathbf{u}_2 - \sin t\,\mathbf{v}_2\Big) + c_3\Big(\sin t\,\mathbf{u}_2 + \cos t\,\mathbf{v}_2\Big).$$

.5pt

d) Eigenvalues/eigenvectors:

$$\lambda = -1: \ \mathbf{v}_1 = (1, -1, 0) \quad , \quad \lambda = 1: \ \mathbf{v}_2 = (0, 0, 1), \ \mathbf{w} = \frac{1}{2}(1, 1, 0).$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-t}\mathbf{v}_1 + c_2 \mathrm{e}^{t}\mathbf{v}_2 + c_3 \mathrm{e}^{t}(t\mathbf{v}_2 + \mathbf{w}).$$

.5pt

e) Eigenvalues/eigenvectors:

$$\lambda = 1: \ \mathbf{v}_1 = (1, 0, 0)\,, \ \mathbf{v}_2 = (0, 1, 0)\,, \ \mathbf{w} = (0, 0, 1).$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{t}\mathbf{v}_1 + c_2 \mathrm{e}^{t}\mathbf{v}_2 + c_3 \mathrm{e}^{t}(t\mathbf{v}_1 + \mathbf{w}).$$

1pt

f) Eigenvalues/eigenvectors:

$$\lambda_1 = 1: \ \mathbf{v}_1 = (1, 0, -1) \quad , \quad \lambda_2 = 1 - \sqrt{2}: \ \mathbf{v}_2 = (1, -\sqrt{2}, 1) \quad , \quad \lambda_3 = 1 + \sqrt{2}: \ \mathbf{v}_3 = (1, \sqrt{2}, 1).$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + c_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2 + c_3 \mathrm{e}^{\lambda_3 t}\mathbf{v}_3.$$

1pt