MAT 475: Practice midterm 1

Exercise 1. For each differential equations, find the equilibria, study their stability and draw the phase line.

a) $x' = x^2 - 4$.

b) $x' = x(x-1)(x-2)$.

c) $x' = \mathrm{e}^{-x^2} - \frac{1}{2}$.

d) $x' = \sin(\cos x)$.

e) $x' = 1 + \sin x$.

Exercise 2. Find f such that $x' = f(x)$ has an stable equilibrium (sink) at $x_* = 1$ and satisfies $f'(1) = 0$.

Exercise 3. Consider the linear system: $\mathbf{x}' = A\mathbf{x}$ with A a 2×2 matrix. Find the general solution in each case:

$$a)\ A = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}, \quad b)\ A = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}, \quad c)\ A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}.$$

Exercise 4. Consider the 4 directional fields in figure 1. Find the associated linear systems $\mathbf{x}' = A\mathbf{x}$ with:

$$a)\ A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad b)\ A = \begin{bmatrix} 0 & \pi/4 \\ 1/2 & 1 \end{bmatrix}, \quad c)\ A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, \quad d)\ A = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}.$$

Exercise 5. Consider the matrices:

$$a)\ A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}, \quad b)\ A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}, \quad c)\ A = \begin{bmatrix} -9 & 8 \\ -2 & -1 \end{bmatrix}.$$

Find the canonical form associated with each matrix, i.e. find T such that $T^{-1}AT$ is of the form:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Sketch the phase portrait of the solutions to $\mathbf{x}' = A\mathbf{x}$ using the basis $\{\mathbf{u}, \mathbf{v}\}$ of your choice.

Exercise 6. Consider the linear system:

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad A = \begin{bmatrix} a & b \\ 1 & a \end{bmatrix}.$$

Find the phase diagram in the ab-plane (i.e. identify the regions where the solutions are spiral, center, node or saddle).

Exercise 7. Consider the harmonic oscillator: $x'' + bx' + kx = 0$ with $b > 0$ and $k > 0$.

a) Write the differential equation as a first order linear system.

b) Find the general solution when $b^2 = 4k$.

c) Find an initial condition $(x(0), x'(0))$ such that the solution $x(t)$ changes sign only once.

Exercise 8. Find the canonical form associated to each matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & -2 \end{bmatrix}.$$

Exercise 9*. Consider the linear system:

$$\mathbf{x}' = A\mathbf{x} \quad \text{, with} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}.$$

Find initial conditions $\mathbf{x}_0$ such that the solution $\mathbf{x}(t)$:

i) converges to zero, i.e. $\mathbf{x}(t) \xrightarrow{t\to+\infty} 0$,

ii) diverges, i.e. $|\mathbf{x}(t)| \xrightarrow{t\to+\infty} +\infty$.

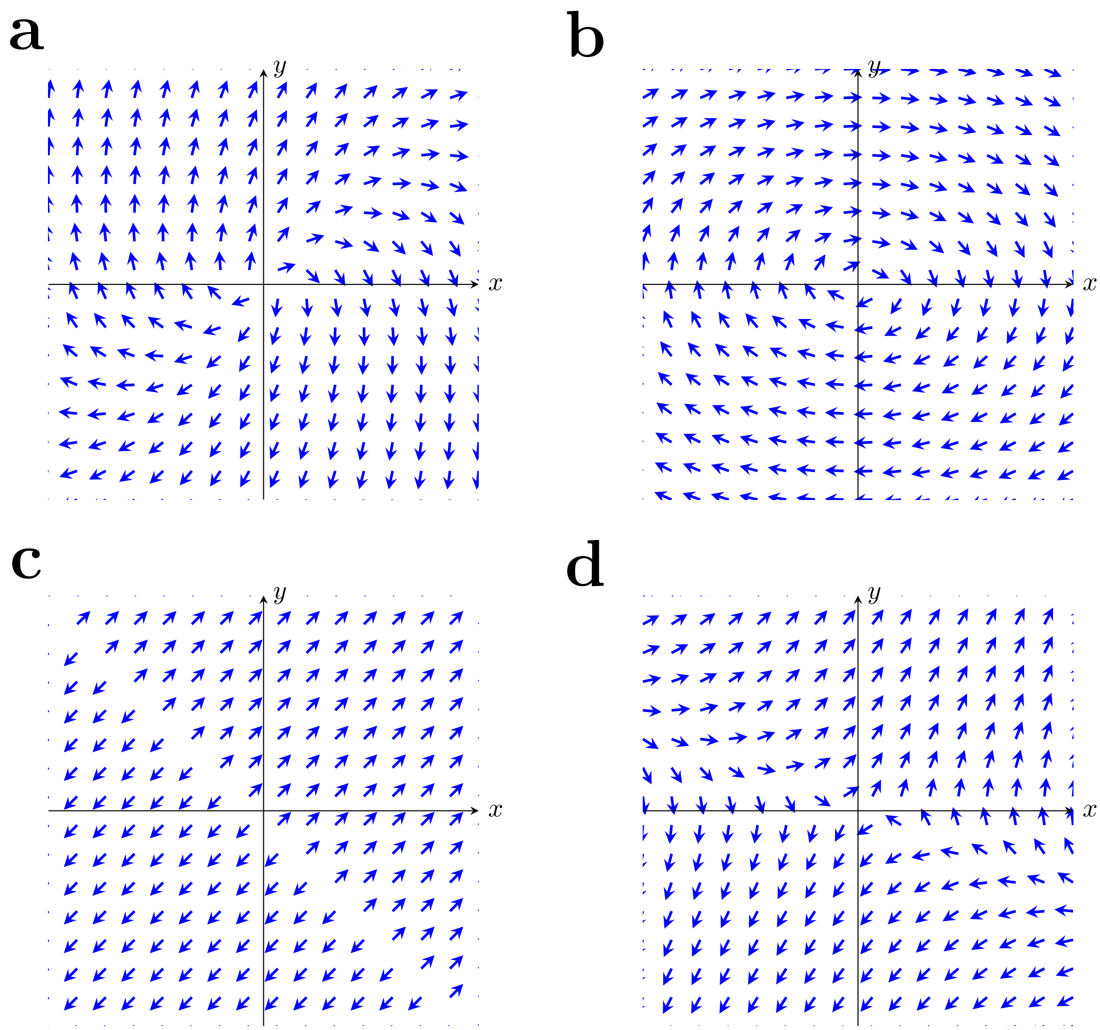

Figure 1: Four direction fields for **Exercise 4**.