MAT 475: Solution Practice midterm 1

Exercise 1.

a) $x' = x^2 - 4$. Equilibria: $x_* = \pm 2$. We have $f'(x) = 2x$, thus $x_* = -2$ is a sink ($f'(-2) < 0$) and $x_* = 2$ is a source ($f'(2) > 0$).

b) $x' = x(x-1)(x-2)$. Equilibria: 0 (source), 1 (sink), 2 (source).

c) $x' = \mathrm{e}^{-x^2} - \frac{1}{2}$. Equilibria: $-\sqrt{\ln 2}$ (source), $\sqrt{\ln 2}$ (sink).

d) $x' = \sin(\cos x)$. Equilibria: $\frac{\pi}{2} + 2k\pi$ (sink), $\frac{\pi}{2} + 2(k+1)\pi$ (source)

e) $x' = 1 + \sin x$. Equilibria: $-\frac{\pi}{2} + 2k\pi$ neither sink or source.

Phase lines are given in figure 1.

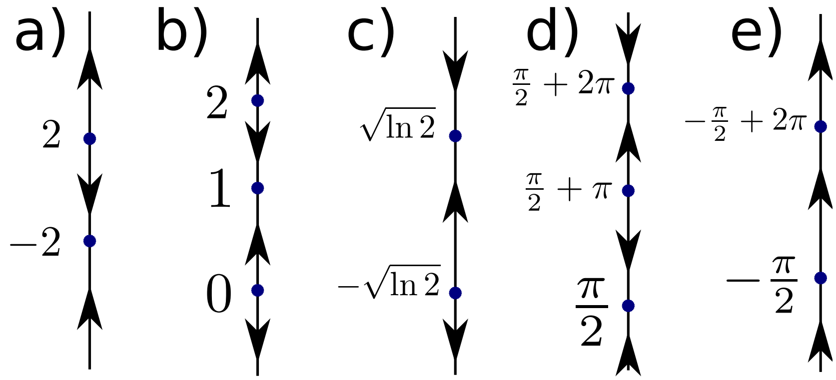

Figure 1: Phase line for Ex. 1.

Exercise 2. Take for example $f(x) = -(x-1)^3$. The equilibrium $x_* = 1$ is stable (draw the phase line) and $f'(1) = 0$.

Exercise 3.

a) Eigenvalues/eigenvectors:

$$\lambda = 1 \pm i, \quad \mathbf{w} = (2, -1-i) = (2,-1) + i(0,-1) = \mathbf{u}_1 + i\mathbf{v}_1.$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^t \Big(c_1(\cos t\,\mathbf{u}_1 - \sin t\,\mathbf{v}_1) + c_2(\sin t\,\mathbf{u}_1 + \cos t\,\mathbf{v}_1) \Big)$$

b) Eigenvalues/eigenvectors:

$$\lambda_1 = -4,\ \mathbf{v}_1 = (1,-1), \qquad \lambda_2 = 2,\ \mathbf{v}_2 = (1,1).$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-4t}\mathbf{v}_1 + c_2 \mathrm{e}^{2t}\mathbf{v}_2.$$

c) Eigenvalues/(generalized) eigenvectors:

$$\lambda = -1,\ \mathbf{v}_1 = (1,1), \quad \lambda = -1,\ (A + \mathrm{Id})\mathbf{w}_1 = \mathbf{v}_1 \Rightarrow \mathbf{w}_1 = (0,1).$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-t}\mathbf{v}_1 + c_2 \mathrm{e}^{-t}(t\mathbf{v}_1 + \mathbf{w}_1).$$

Exercise 4.

a) Corresponds to figure **c**. All the vectors are colinear to $(1,1)$.

b) It is a saddle point ($D = \det(A) < 0$). It corresponds to figure **d**.

c) $D = \det(A) = 10$ and $T = \mathrm{trace}(A) = 5$. Since $D > T^2/4$, solutions are spiral. Thus, figure **a**.

d) $D = \det(A) > 0$ and $T = \mathrm{trace}(A) = 0$. Solutions are center: figure **b**.

Exercise 5.

a) Eigenvalues/eigenvectors:

$$\lambda_1 = 1,\ \mathbf{v}_1 = (1,1), \quad \lambda_2 = 3,\ \mathbf{v}_2 = (0,1).$$

Taking $T = [\mathbf{v}_1,\ \mathbf{v}_2]$, we obtain the canonical form:

$$T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

We give the phase portrait in figure 2 using as a basis $\{\mathbf{v}_1,\ \mathbf{v}_2\}$.

b) Eigenvalues/eigenvectors:

$$\lambda = -1 \pm 2i,\ \mathbf{w} = (-1 \pm i, 2) = (-1,2) + i(1,0) = \mathbf{u}_1 + i\mathbf{v}_1.$$

Taking $T = [\mathbf{u}_1,\ \mathbf{v}_1]$, we obtain the canonical form:

$$T^{-1}AT = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}.$$

We give the phase portrait in figure 2 using as a basis $\{\mathbf{u}_1,\ \mathbf{v}_1\}$.

c) Eigenvalues/(generalized) eigenvectors:

$$\lambda = -5,\ \mathbf{v}_1 = (2,1), \quad \lambda = -5,\ (A + 5\mathrm{Id})\mathbf{w}_1 = \mathbf{v}_1 \Rightarrow \mathbf{w}_1 = (-1/2, 0).$$

Taking $T = [\mathbf{u}_1,\ \mathbf{v}_1]$, we obtain the canonical form:

$$T^{-1}AT = \begin{bmatrix} -5 & 1 \\ 0 & -5 \end{bmatrix}.$$

We give the phase portrait in figure 2 using as a basis $\{\mathbf{v}_1,\ \mathbf{w}_1\}$.

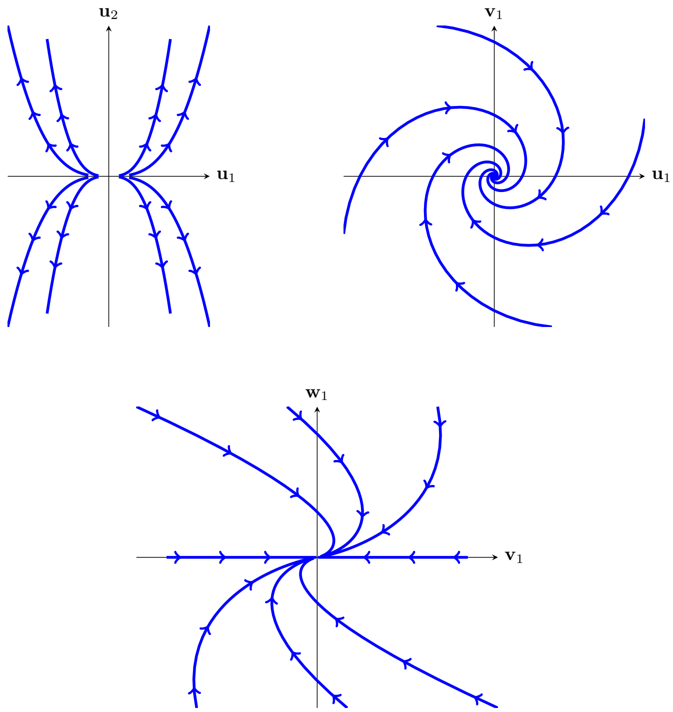

Figure 2: Phase portrait for Ex. 5: top-left for A, top-right for B and bottom for C.

Exercise 6. We use the determinant and trace of A to determine the relevant region. Here,

$$D = \det(A) = a^2 - b \quad , \quad T = \operatorname{trace}(A) = 2a.$$

The saddle points are in the region $D < 0 \Rightarrow b > a^2$. To determine the 'spiral region', we study: $D > T^2/4 \Rightarrow a^2 - b > a^2 \Rightarrow b < 0$. Thus, nodes are in the region $0 < b < a^2$. Finally, the centers are along the line $T = 0$ and $D > 0$ which gives: $a = 0$ and $b > 0$.

The phase diagram is given in figure 3

Exercise 7.

a) $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}$.

b) Eigenvalues/(generalized) eigenvectors:

$$\lambda = -\frac{b}{2}, \, \mathbf{v}_1 = (1, -b/2), \quad \lambda = -\frac{b}{2}, \, \left(A + \frac{b}{2}\mathrm{Id}\right)\mathbf{w}_1 = \mathbf{v}_1 \Rightarrow \mathbf{w}_1 = (0, 1).$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-\frac{b}{2}t}\mathbf{v}_1 + c_2 \mathrm{e}^{-\frac{b}{2}t}(t\mathbf{v}_1 + \mathbf{w}_1) = \mathrm{e}^{-\frac{b}{2}t} \begin{pmatrix} c_1 + c_2 t \\ -\frac{b}{2}c_1 + -\frac{b}{2}c_2 t + c_2 \end{pmatrix}.$$

c) In order to have $x(t)$ changing sign once, we can take $c_1 = 1$ and $c_2 = -1$. We obtain: $x(t) = \mathrm{e}^{-\frac{b}{2}t}(1 - t)$ which is positive for $t \in [0, 1]$ and then negative for $t \geq 1$. This choice corresponds to $x(0) = 1$, $x'(0) = -b/2 - 1$.

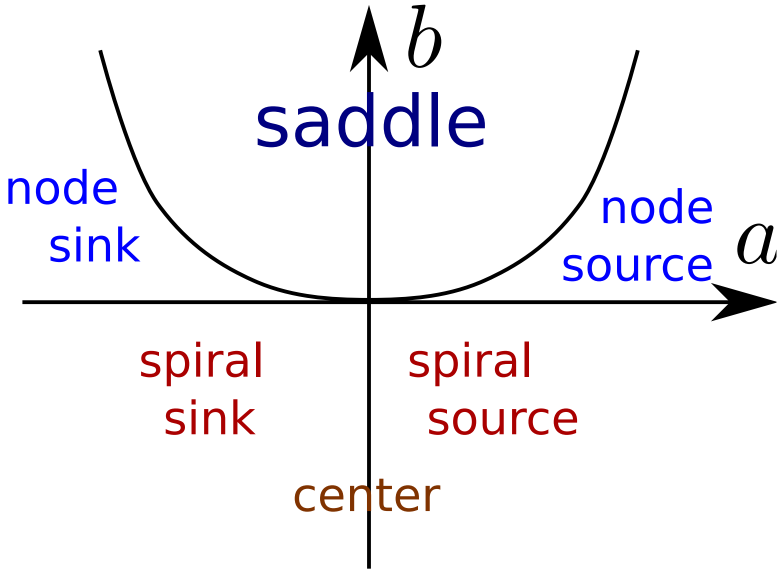

Figure 3: Phase diagram for **Ex.** 6.

Exercise 8.

The matrix A has real and distinct eigenvalues $\lambda = 1,\ 2,\ 3$, thus:

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

where $\sim$ means the two matrices are similar, i.e. $A \sim D$ if there exists T such that $A = TDT^{-1}$.

The eigenvalues of B are $\lambda = 2,\ 1 \pm i$, thus its canonical form has two "blocks":

$$B \sim \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & -1 & 1 \end{array}\right].$$

The matrix C has a repeated eigenvalue $\lambda = 1$ with only one eigenvector and $\lambda = 2$ as eigenvalue, thus:

$$C \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right].$$

Exercise 9.

The characteristic polynomial of A is given by:

$$P_A(\lambda) = -\lambda(\lambda^2 - 2) - 2(-\lambda) = -\lambda(\lambda^2 - 4) = -\lambda(\lambda - 2)(\lambda + 2).$$

Thus, there are 3 eigenvalues: $\lambda = -2,\ 0,\ 2$.

In order to have a solution converging to zero, we need to find the eigenvector associated with the eigenvalue $\lambda = -2$:

$$(A + 2\mathrm{Id})u = 0 \ \Rightarrow \ \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 2 \end{bmatrix} u = 0 \ \Rightarrow \ u = (-1, 2, -1).$$

Taking $\mathbf{x}_0 = (-1, 2, -1)$, we find that the solution will satisfy $\mathbf{x}(t) \overset{t\to+\infty}{\longrightarrow} 0$.

Similarly, to find a divergent solution, we look for the eigenvector associated with $\lambda = 2$:

$$(A - 2\mathrm{Id})u = 0 \;\Rightarrow\; \begin{bmatrix} -2 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & -2 \end{bmatrix} u = 0 \;\Rightarrow\; u = (1, 2, 1).$$

Let $\mathbf{x}_0 = (1, 2, 1)$, the solution will satisfy $|\mathbf{x}(t)| \overset{t\to+\infty}{\longrightarrow} +\infty$.

Remark. Without any computation, we could have chosen $\mathbf{x}_0 = (0, 0, 0)$ to find a solution that converges to zero. Moreover, unless we are very (very) unlucky, a random initial condition (i.e. $\mathbf{x}_0 = (\pi, \mathrm{e}, \ln 2)$) will give a solution that satisfy $|\mathbf{x}(t)| \overset{t\to+\infty}{\longrightarrow} +\infty$.