Weak convergence in Hilbert space

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We suppose that H is a Hilbert space and denote by $\langle , \rangle$ and $\|.\|$ its inner product and norm (respectively).

Definition. We say that a sequence f_n converges weakly to f in H if it satisfies:

$$\langle f_n, g \rangle \stackrel{n \to +\infty}{\longrightarrow} \langle f, g \rangle \qquad \text{for all } g \in H.$$

We note $f_n \stackrel{H}{\rightharpoonup} f$.

Example. Consider $H = L^2(\mathbb{R})$ with

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x)\, dx \qquad \|f\| = \langle f, f \rangle^{1/2} = \left(\int_{\mathbb{R}} |f(x)|^2\, dx \right)^{1/2}.$$

Let $f \in L^2(\mathbb{R})$ and $f \neq 0$. Define the sequence (see figure 1):

$$f_n(x) = f(x - n).$$

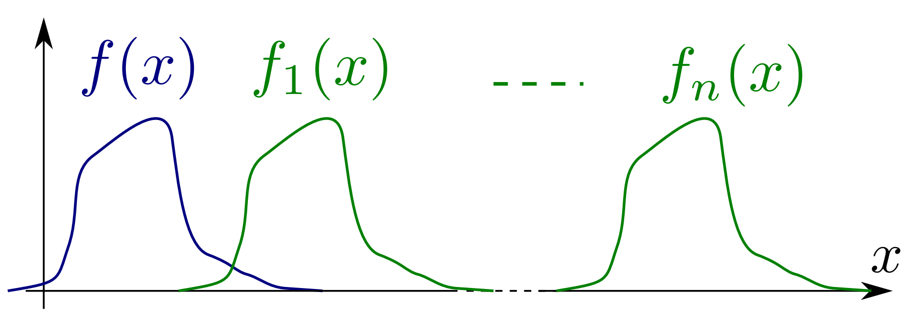

Figure 1: A sequence of functions f_n that converges weakly to zero but not strongly.

The sequence converges $\{f_n\}_n$ **weakly** to zero but **not strongly** in $L^2(\mathbb{R})$:

$$f_n \rightharpoonup 0 \qquad \text{and} \qquad f_n \nrightarrow 0.$$

Indeed, $\|f_n - 0\| = \|f_n\| = \|f\| \neq 0$, thus the sequence cannot converge to zero strongly.

Now to show that it converges weakly to zero, we take $g \in L^2(\mathbb{R})$ and study:

$$\langle f_n, g \rangle = \int_{\mathbb{R}} f(x-n) g(x)\, dx.$$

Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we can find $\overline{f}$ and $\overline{g}$ with compact support such that:

$$\|f - \overline{f}\| \leq \varepsilon \qquad \text{and} \qquad \|g - \overline{g}\| \leq \varepsilon.$$

It is clear that since $\overline{f}$ and $\overline{g}$ have compact support, we have $\overline{f}(x-n)\overline{g}(x) = 0$ for n large enough since the support of the functions will not intersect (the support of $\overline{f}(x-n)$ *"goes to"* $+\infty$). Thus,

$$\int_{\mathbb{R}} \overline{f}(x-n)\overline{g}(x)\, dx = 0 \qquad \text{for } n \text{ large enough.}$$

We deduce:

$$\begin{array}{rcccccc} \langle f_n, g \rangle & = & \langle f_n - \overline{f}_n, g \rangle & + & \langle \overline{f}_n, g - \overline{g} \rangle & + & \langle \overline{f}_n, \overline{g} \rangle \\ \Rightarrow |\langle f_n, g \rangle| & \leq & \|f_n - \overline{f}_n\| \cdot \|g\| & + & \|\overline{f}_n\| \cdot \|g - \overline{g}\| & + & |\langle \overline{f}_n, \overline{g} \rangle| \\ & \leq & \varepsilon \|g\| & + & (\|f\| + \varepsilon) \cdot \varepsilon & + & 0, \end{array}$$

for n large enough. Since ε is arbitrary whereas f and g are fixed, we deduce that $\langle f_n, g \rangle \overset{n \to +\infty}{\longrightarrow} 0$. $\square$

A fundamental property of the weak convergence is that one can extract from any bounded sequence $\{f_n\}$ a subsequence $\{f_{n_k}\}$ that will converge weakly. In other words, bounded sets become compact sets for the weak topology. Let's prove this result.

Theorem 1 *Suppose $\{f_n\}_n$ bounded sequence in H Hilbert. Then there exists a subsequence n_k and f in H such that:*

$$f_{n_k} \rightharpoonup f.$$

Proof. Denote C the constant such that $\|f_n\| \leq C$ and consider a Hilbert basis $\{\mathbf{e_n}\}_{n \geq 1}$ of H.

The sequence $\langle f_n, \mathbf{e_1} \rangle$ is bounded in $\mathbb{R}$:

$$|\langle f_n, \mathbf{e_1} \rangle| \leq \|f_n\| \cdot \|\mathbf{e_1}\| \leq C.$$

Thus, the sequence $x_n^1 = \langle f_n, \mathbf{e_1} \rangle$ is in $[-C, C]$ compact of $\mathbb{R}$. By Bolzano-Weierstrass, there exists a subsequence n_k^1 and a scalar x_1 such that:

$$x_{n_k^1} \overset{k \to +\infty}{\longrightarrow} x_1 \quad \text{i.e. } \langle f_{n_k^1}, \mathbf{e_1} \rangle \overset{k \to +\infty}{\longrightarrow} x_1.$$

Similarly, the sequence $x_k^2 = \langle f_{n_k^1}, \mathbf{e_2} \rangle$ is bounded in $\mathbb{R}$ ($x_k^2 \in [-C, C]$ for any k). Thus, we can extract a subsequence n_k^2 from n_k^1 and find a scalar x_2 such that:

$$x_{n_k^2} \overset{k\to+\infty}{\longrightarrow} x_2 \quad \text{i.e. } \langle f_{n_k^2}, \mathbf{e_2} \rangle \overset{k\to+\infty}{\longrightarrow} x_2.$$

Notice that we sill have $\langle f_{n_k^2}, \mathbf{e_1} \rangle \overset{k\to+\infty}{\longrightarrow} x_1$. Iteratively, we construct subsequences n_k^i such that: n_k^{i+1} subsequence of n_k^i and

$$\langle f_{n_k^i}, \mathbf{e_i} \rangle \overset{k\to+\infty}{\longrightarrow} x_i.$$

Consider now the subsequence $n_k = n_k^k$ (Cantor's diagonal argument) and $f = \sum_{i=1}^{+\infty} x_i \, \mathbf{e_i}$.
Claim: $f_{n_k^k}$ converges weakly to f.

To prove this result, we first need to show that f is in H (the coefficients x_i might 'explode'). For any $\mathbf{e_i}$, we have:

$$|\langle f, \mathbf{e_i} \rangle| = |x_i| = \lim_{k\to+\infty} |\langle f_{n_k^k}, \mathbf{e_i} \rangle| \leq \lim_{k\to+\infty} C \cdot \|\mathbf{e_i}\| = C.$$

By density, we deduce that: $|\langle f, \varphi \rangle| \leq C\|\varphi\|$ for any φ in H. Therefore, $\|f\| \leq C$ and f is well defined.

Moreover, using a similar argument, we have:

$$\lim_{k\to+\infty} \langle f - f_{n_k^k}, \mathbf{e_i} \rangle = x_i - \lim_{k\to+\infty} \langle f_{n_k^k}, \mathbf{e_i} \rangle = x_i - x_i = 0.$$

By density, we deduce that for any φ in H:

$$\lim_{k\to+\infty} \langle f - f_{n_k^k}, \varphi \rangle = 0.$$

Thus, $f_{n_k^k} \rightharpoonup f$. □

Remark 0.1 *In the proof, we have used that there exists a Hilbert basis for H. This is true for $L^2(\Omega)$ since it is a separable space. However, one can skip this assumption. Indeed, rather than consider H, we can just consider the vector space V spanned by the sequence $\{f_n\}_n$:*

$$\begin{aligned} V &= \mathit{Span}(f_1, f_2, \ldots, f_n, \ldots) \\ &= \{f \in H \,|\, f = c_1 f_1 + \ldots + c_n f_n\} \quad \textit{(all the finite linear combinations)} \end{aligned}$$

The closure of V in H, denoted $\overline{V}$, is itself a Hilbert space and it is separable since the sequence $\{f_n\}_n$ is dense in it. Thus, if H is not separable, we can apply the argument of the proof to $\overline{V}$ rather than H to extract a subsequence that converges weakly.