APM 576: Homework 2 (09/27)

1 L^p and Hilbert spaces

Ex 1.

Consider the constant function $f = 1$, it is a function of $L^\infty(\mathbb{R})$ with $\|f\|_\infty = 1$. We claim that no sequence in $C_c^0(\mathbb{R})$ can approach f. Indeed, take any $\varphi \in C_c^0(\mathbb{R})$ and denotes K its support. Since K is compact, it is contained in an interval of the form $[-M, M]$. Thus, $\varphi(x) = 0$ for $|x| > M$. Thus,

$$\|f - \varphi\|_\infty \geq \sup_{|x| \geq M} |f(x) - \varphi(x)| = 1.$$

Thus, $\|f - \varphi\|_\infty \geq 1$. Therefore, we cannot have $\varphi_n \to f$ in $L^\infty(\mathbb{R})$ with $\varphi_n \in C_c^0(\mathbb{R})$.

Ex 2.

Let H be a Hilbert space, $\ell \in H'$ and $V = \mathrm{Ker}(\ell)$.

a) If $V = H$, this means $\ell = 0$ and we can take $u_\ell = 0$ as a representative.

b) If $V \neq H$, denote $u_0 \notin V$ and p_0 its projection on V.
Consider the vector $b = u_0 - p_0$. If one considers $\ell_b(v) = \langle v, u_1 \rangle$, we observe that ℓ_b as the same kernel as ℓ:

$$\ell(v) = 0 \quad \Leftrightarrow \quad v \in H \quad \Leftrightarrow \quad v \perp u_1 \quad \Leftrightarrow \quad \ell_b(v) = 0.$$

But ℓ and ℓ_b do not necessarily agree on the *line* b. We just have to normalize. Let $\alpha = \ell(b)/\|b\|^2$ ($\alpha \neq 0$ since $b \notin H$) and consider $u_\ell = \alpha\, b$. We claim that $\ell_{u_\ell}(v) = \langle u_\ell, v \rangle$ is equal to ℓ.
Indeed, ℓ_{u_ℓ} and ℓ are both equal to zero on H. Moreover:

$$\ell_{u_\ell}(b) = \langle u_\ell, b \rangle = \langle \alpha b, b \rangle = \langle \ell(b)/\|b\|^2 b, b \rangle = \ell(b).$$

By linearity, ℓ_{u_ℓ} and ℓ agrees on $V \oplus b = H$. Therefore, $\ell_{u_\ell} = \ell$.

c) Suppose that there exists another v_ℓ such that $\ell(v) = \langle v, v_\ell \rangle$ for all $v \in H$. By linearity, we would have:

$$\langle v, v_\ell - u_\ell \rangle = 0 \qquad \text{for all } v \in H.$$

Taking $v = v_\ell - u_\ell$, we find $\|v_\ell - u_\ell\|^2 = 0$ therefore $v_\ell = u_\ell$.

2 Sobolev spaces

Ex 3.

The function u is a *pyramid*. We divide the set Ω in four regions (see Fig.):

$$\begin{aligned}
\Omega_1 &= \{0 < x_1 < 1,\, |x_2| < x_1\},\\
\Omega_2 &= \{-1 < x_1 < 0,\, |x_2| < -x_1\},\\
\Omega_3 &= \{0 < x_2 < 1,\, |x_1| < x_2\},\\
\Omega_4 &= \{-1 < x_2 < 0,\, |x_1| < -x_2\}.
\end{aligned}$$

Notice that the four sets do not cover exactly Ω (i.e. $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \neq \Omega$) as we miss the two diagonals (i.e. $x_2 = \pm x_1$). But the missing diagonals are a negligeable region in $\mathbb{R}^2$, thus it will not affect the integral (since the functions will be locally integrable).

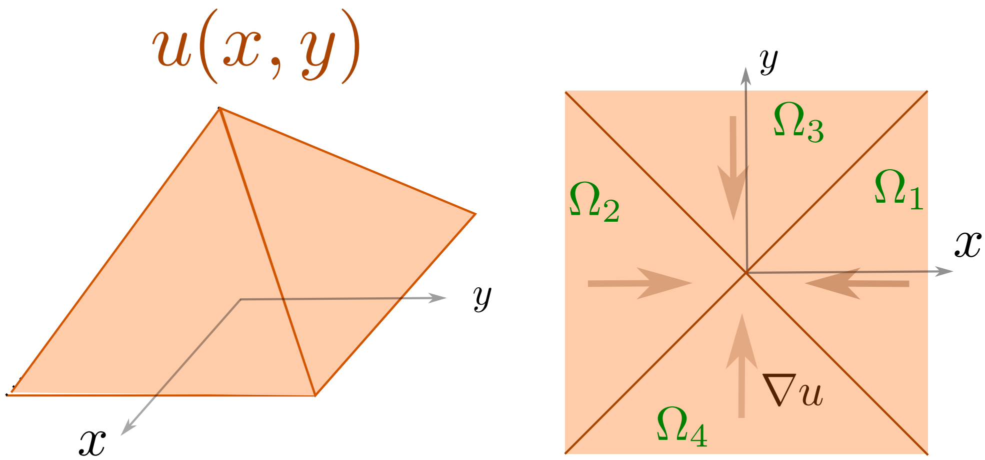

Figure 1: The function u (left) and its gradient ∇u (right) *outside* the diagonal $y = \pm x$.

It is clear that u as *classical* derivative on each region Ω_i:

$$\nabla u(x_1, x_2) = \begin{cases} (-1, 0)^T & \text{on } \Omega_1 \\ (1, 0)^T & \text{on } \Omega_2 \\ (0, -1)^T & \text{on } \Omega_3 \\ (0, 1)^T & \text{on } \Omega_4 \end{cases}$$

Thus, we can write $\partial_{x_1} u$ and $\partial_{x_2} u$ inside each region Ω_i.

We have now to show that there are indeed the weak derivative of u on the all domain Ω (i.e. show no Dirac appearing on the diagonals). Take $\varphi \in C_c^\infty(\Omega)$:

$$\int_\Omega u \partial_{x_1} \varphi \, \mathrm{d}x_1 \mathrm{d}x_2 \quad = \quad \int_{\Omega_1} \ldots + \int_{\Omega_2} \ldots + \int_{\Omega_3} \ldots + \int_{\Omega_4} \ldots$$

Moreover,

$$\begin{aligned}
\int_{\Omega_1} u\partial_{x_1}\varphi \,\mathrm{d}x_1\mathrm{d}x_2 &= \int_{x_2=-1}^{x_2=1}\int_{x_1=|x_2|}^{x_1=1} u\partial_{x_1}\varphi \,\mathrm{d}x_1\mathrm{d}x_2 \\
&= \int_{x_2=-1}^{x_2=1}\int_{x_1=|x_2|}^{x_1=1} -\partial_{x_1}u\,\varphi \,\mathrm{d}x_1\mathrm{d}x_2 \\
&\quad + \int_{x_2=-1}^{x_2=1} [u(1,x_2)\varphi(1,x_2) - u(|x_2|,x_2)\varphi(|x_2|,x_2)]\,\mathrm{d}x_2 \\
&= -\int_{\Omega_1} \partial_{x_1}u\,\varphi \,\mathrm{d}x_1\mathrm{d}x_2 - \int_{x_2=-1}^{x_2=1} u(|x_2|,x_2)\varphi(|x_2|,x_2)\,\mathrm{d}x_2,
\end{aligned}$$

using that $\varphi = 0$ on the boundary of Ω. We proceed similarly for the domain Ω_2 and obtain:

$$\int_{\Omega_2} u\partial_{x_1}\varphi \,\mathrm{d}x_1\mathrm{d}x_2 = -\int_{\Omega_2} \partial_{x_1}u\,\varphi \,\mathrm{d}x_1\mathrm{d}x_2 + \int_{x_1=-1}^{x_1=1} u(x_1,-|x_1|)\varphi(x_1,-|x_1|)\,\mathrm{d}x_1.$$

We observe that the integral along the diagonals cancel out once we regroup the four integral. Thus,

$$\int_{\Omega} u\partial_{x_1}\varphi \,\mathrm{d}x_1\mathrm{d}x_2 = -\sum_{i=1\ldots 4}\int_{\Omega_i} \partial_{x_1}u\,\varphi \,\mathrm{d}x_1\mathrm{d}x_2 \quad +0 = -\int_{\Omega} \partial_{x_1}u\,\varphi \,\mathrm{d}x_1\mathrm{d}x_2.$$

Thus, $\partial_{x_1}u$ is indeed the weak derivative of u. We can proceed similarly for $\partial_{x_2}u$.

To conclude, both functions $\partial_{x_1}u$ and $\partial_{x_2}u$ are in $L^\infty(\Omega)$. Therefore, there are also in $L^p(\Omega)$ for any p. We deduce that $u \in W^{1,p}(\Omega)$ for all $1 \leq p \leq +\infty$.

3 Approximation

Ex 4.

Denote $\overline{V}$ the closure of V. By assumption $\overline{V}$ is a compact and strictly included in U. Let

$$d = \operatorname{dist}(\overline{V}, U^C) = \inf_{(x,y)\in\overline{V}\times U^C}\{|x-y|\}. \tag{1}$$

Since $\overline{V}$ is a compact and U^C a closed set, the distance is actually reached (i.e. the *inf* in eq. (1) is a *min*). Moreover, $\overline{V}\cap U^C = \emptyset$, thus $d > 0$. Consider then:

$$W = \{x \in U \,|\, \operatorname{dist}(x, \overline{V}) < d/2\}.$$

The set W is in between V and U. We now have to smooth the indicator function $\mathbb{1}_W$ to conclude.

Consider the usual mollifier function:

$$\eta(x) = \begin{cases} c\,\mathrm{e}^{\frac{1}{|x|-1}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

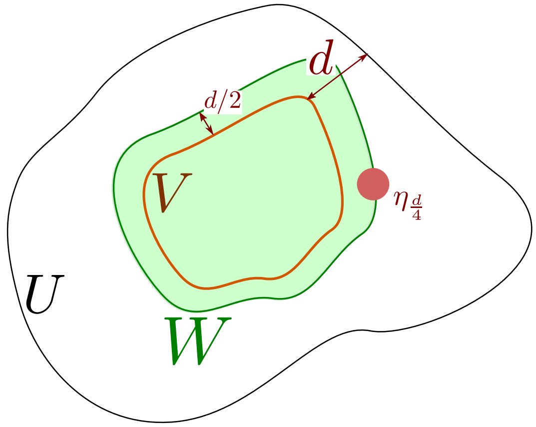

Figure 2: The set W in between V and U. The characteristic function $\mathbb{1}_W$ is then smoothed with a mollifier function η with a *small enough* support (of size $d/4$).

where $c > 0$ is a constant such that $\int_{\mathbb{R}^n} \eta(x)\,\mathrm{d}x = 1$. The function n is C^∞ and its support is the unit ball. To *shrink* its support, we proceed as follow:

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta(x/\varepsilon).$$

The function η_ε is still C^∞ and satisfies $\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,\mathrm{d}x = 1$, but moreover its support is the ball of radius ε.

To conclude, we consider $\zeta = \mathbb{1}_W * \eta_\varepsilon$ with $\varepsilon = d/4$.

Ex 5.

Assume U is bounded and $U \subset\subset \cup_{i=1}^N V_i$ (see Fig. 3). Consider a collection of smaller open sets W_i such that:

$$\overline{W}_i \subset V_i \quad \text{and} \quad U \subset\subset \cup_{i=1}^N W_i.$$

See remark below and fig. 3 on how to obtain such open sets W_i.

From exercise 4, we can find C^∞ functions h_i such that:

$$h_i = 1 \ \text{ on } W_i \quad \text{and} \quad h_i = 0 \ \text{ on } V_i^C.$$

We now construct iteratively the functions:

$$\begin{aligned} \zeta_1 &= h_1 \\ \zeta_2 &= (1-h_1)h_2 \\ \dots & \quad \dots \\ \zeta_N &= (1-h_1)(1-h_2)\dots(1-h_{N-1})h_N. \end{aligned}$$

Notice that $\mathrm{Supp}(\zeta_i) \subset \mathrm{Supp}(h_i) \subset V_i$ and $0 \le \zeta_i \le 1$. Moreover, by induction, one can show that:

$$\sum_{i=1}^N \zeta_i(x) = 1 - (1-h_1)(1-h_2)\dots(1-h_N).$$

In particular, for in $x \in U$, there exists W_i such that $x \in W_i$ and therefore $h_i(x) = 1$ and:

$$\sum_{i=1}^{N} \zeta_i(x) = 1.$$

Thus, $\sum_{i=1}^{N} \zeta_i(x) = 1$ on U.

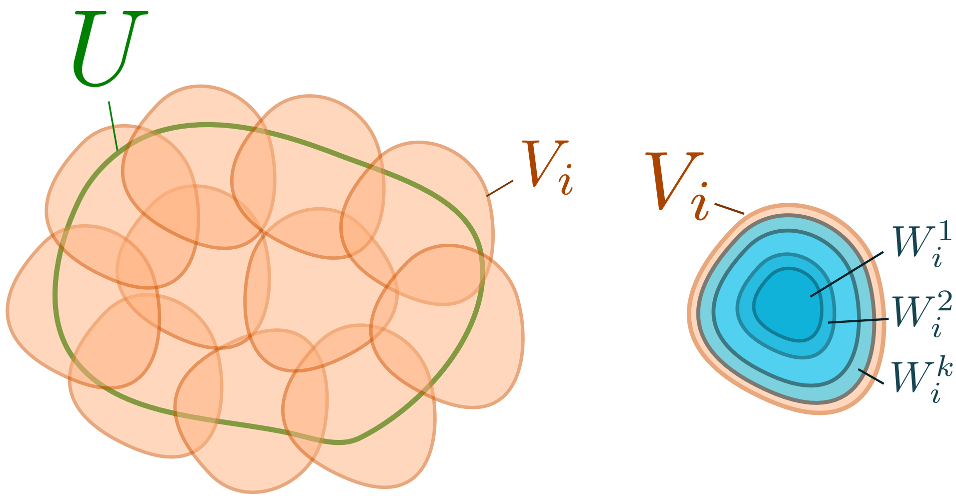

Figure 3: **Left**: the set $\overline{U}$ is covered by a union of open sets $\cup_{i=1}^{N} V_i$. **Right**: approximate V_i by an increase sequence of open sets W_i^k (i.e. *onion* approx.)

Remark. To find the open sets W_i such that $\overline{W_i} \subset V_i$ and $U \subset\subset \cup_{i=1}^{N} W_i$, consider for each i:

$$W_i^k = \{x \in V_i \mid \text{dist}(x, V_i^C) < \frac{1}{k}\}. \tag{2}$$

The sets W_i^k looks like a *onion* and satisfies $W_i^k \subset W_i^{k'}$ for $k \leq k'$. We can check that $\cup_{k=1}^{\infty} W_i^k = V_i$. Thus,

$$\overline{U} \subset \bigcup_{i=1}^{N} \bigcup_{k=1}^{\infty} W_i^k.$$

Since $\overline{U}$ is compact, one can extract a *finite* subset that covers $\overline{U}$:

$$\overline{U} \subset \bigcup_{j=1}^{M} W_{i_j}^{k_j}.$$

Consider k_* the maximum of k_j: $k_* = \max_{j=1...M} k_j$. Since the sequence W_i^k is increasing with respect to k, we find:

$$\bigcup_{i=1}^{N} W_i^{k_*} \quad \supset \quad \bigcup_{j=1}^{M} W_{i_j}^{k_j} \quad \supset \quad \overline{U}.$$

The family of sets $W_i^{k_*}$ satisfies all the conditions.

4 Trace

Ex 6.

We consider the 2D case and take $\Omega = B(\mathbf{0}, 1) = \{x^2 + y^2 < 1\}$. Fix p such that $1 \leq p < \infty$. We would like to construct functions f such that the values at the boundary $\partial\Omega$ is *large* whereas its L^p norm is *small*. The key is to take functions that *vanish* inside Ω and *explode* near $\partial\Omega$.

Define the functions:

$$f_k(x, y) = k^{1/p}(x^2 + y^2)^k. \tag{3}$$

For any k, the function f_k is continuous and bounded (therefore in $L^p(\Omega)$ for any $1 \leq p \leq \infty$). Notice that at the boundary $\partial\Omega$, we have:

$$f_{k|\partial\Omega} = k^{1/p} \quad \overset{k\to+\infty}{\longrightarrow} +\infty.$$

Moreover,

$$\int_\Omega f_k^p = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} kr^{2kp}\,\mathrm{d}r\mathrm{d}\theta = k2\pi\,\frac{r^{2kp+1}}{2kp+1}\Big|_0^1 = \frac{2\pi k}{2kp+1} < \frac{\pi}{p}.$$

Therefore $\|f_k\|_p \leq C$. Therefore, the trace does not define a continuous application from $L^p(\Omega)$ to $L^p(\partial\Omega)$.

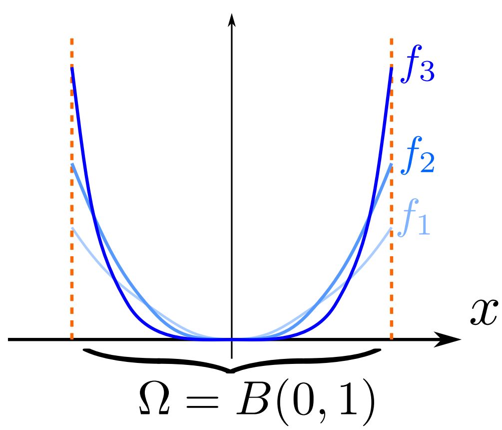

Figure 4: Example of functions f_k (see eq. (3)) with bounded norm in L^p but a trace that goes to infinity.

5 Inequalities

Ex 7.

Consider first $u \in C_c^\infty(\Omega)$:

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &= \sum_i \int_\Omega |\partial_{x_i} u|^2 = \sum_i \int_\Omega \partial_{x_i} u \cdot \partial_{x_i} u \\ &= \sum_i \left(-\int_\Omega \partial_{x_i}^2 u \cdot u \ + \ \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \cdot u \right) \\ &= -\int_\Omega \Delta u \cdot u \ + \ 0\end{aligned}$$

since $u = 0$ on $\partial\Omega$ (or if Ω unbounded using that u has compact support). Using Cauchy-Schwarz inequality, we deduce:

$$\|\nabla u\|_{L^2}^2 = -\int_\Omega \Delta u \cdot u \le \left(\int_\Omega (\Delta u)^2 \right)^{1/2} \left(\int_\Omega u^2 \right)^{1/2}.$$

Moreover:

$$(\Delta u)^2 = \left(\sum_i \partial_{ii}^2 u \right)^2 \le d \sum_i (\partial_{ii}^2 u)^2 \le d \|D^2 u\|^2,$$

where d is the space dimension[1] Thus,

$$\|\nabla u\|_{L^2} \le C \|D^2 u\|_{L^2}^{1/2} \ \|u\|_{L^2}^{1/2}, \tag{4}$$

with $C = d^{1/4}$.

We now extend this result for $u \in C^2(\Omega) \cap H_0^1(\Omega)$. The only point that needs to be addressed is the justification of the integration by parts: $\frac{\partial u}{\partial \eta} u$ does not necessarily make sense.

Denote $\{v_k\}_k \subset C_c^\infty(\Omega)$ converging to u in $H_0^1(\Omega)$ and $\{w_k\}_k \subset C_c^\infty(\overline{\Omega})$ converging to u in $H^2(\Omega)$. Note that if $u \notin H^2(\Omega)$, then $\|D^2 u\|_{L^2} = \infty$ and the inequality (4) is satisfied. Assuming $u \in H^2(\Omega)$, we have:

$$\begin{aligned}\|\nabla u\|_{L^2}^2 &= \int_\Omega \nabla u \cdot \nabla u = \lim_{k \to \infty} \int_\Omega \nabla v_k \cdot \nabla w_k = \lim_{k \to \infty} \left(\int_\Omega -v_k \cdot \Delta w_k + \int_{\partial\Omega} v_k \frac{\partial w_k}{\partial \eta} \right) \\ &= \lim_{k \to \infty} \int_\Omega -v_k \cdot \Delta w_k + 0 = \int_\Omega -u \cdot \Delta u,\end{aligned}$$

since $v_k = 0$ on $\partial\Omega$. We can then apply the same argument as before to conclude.

Ex 8.

Suppose Ω connected and $u \in W^{1,p}(\Omega)$ satisfies:

$$\nabla u = 0 \quad \text{a.e. in } \Omega.$$

[1] $(\sum_{i=1}^n a_i)^2 \le n \sum_{i=1}^n a_i^2$ by Cauchy-Schwarz inequality (again).

Take a point $x_0 \in \Omega$ and consider a neighborhood $\mathcal{O}$ of x_0 in Ω such that $\mathcal{O} \subset\subset \Omega$ (for example the ball $B(x_0, r)$ with $r > 0$ small enough). Let $d = \text{dist}(\overline{\mathcal{O}}, \Omega^c) > 0$. For $\varepsilon < d$, the smooth function $u * \eta_\varepsilon$ is well-defined on $\mathcal{O}$ and satisfies:

$$\nabla(u * \eta_\varepsilon) = \nabla u * \eta_\varepsilon = 0.$$

Therefore $u * \eta_\varepsilon$ is constant function on $\mathcal{O}$. The limit as $\varepsilon \to 0$ has to be constant as well, thus u constant on $\mathcal{O}$. In particular, u is continuous at x_0. Since x_0 can be any point in Ω, u is continuous on Ω and locally constant.

To conclude, consider the domain $A = \{x \in \Omega \mid u(x) = u(x_0)\}$. The domain A is non-empty ($x_0 \in A$), close (u continuous) and open (since u locally constant). Since Ω is connected, we must $A = \Omega$ and therefore u constant on Ω.