APM 576: Homework 3 (10/23)

1 Boundary conditions

Ex 1.

Assume Ω is connected and $f \in L^2(\Omega)$. A function $u \in H^1(\Omega)$ is a weak solution of *Neumann's problem*:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where η is outward unit normal vector on the boundary Ω, if for all $v \in H^1(\Omega)$:

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x. \tag{2}$$

Notice that we do not suppose $v = 0$ at $\partial\Omega$ (i.e. $v \notin H_0^1(\Omega)$).

a) Show that a *necessary* condition to have a weak solution is f satisfying:

$$\int_\Omega f \,\mathrm{d}x = 0. \tag{3}$$

b) We try to show that (3) is also a *sufficient* condition to have a weak solution.

Consider E the subspace of $H^1(\Omega)$ with zero mean:

$$E = \{u \in H^1(\Omega) \mid \int_\Omega u \,\mathrm{d}x = 0\}.$$

Apply Lax-Milgram theorem to show existence and uniqueness of solution to the weak problem (2) **on E**.
Does the weak problem (2) have a unique solution in $H^1(\Omega)$?

Remark. *(Hint for Ex. 3)* The condition (3) is sometimes referred to as the *compatibility* condition. Notice that another way to write this condition is $\langle f, \mathbf{1}\rangle_{L^2(\Omega)} = 0$ where $\mathbf{1}$ is the constant function on Ω. Moreover:

$$E = \{\mathbf{1}\}^\perp.$$

Notice also that $\mathbf{1}$ is in the kernel of the operator $Lu = -\Delta u$ with boundary conditions $\frac{\partial u}{\partial \eta} = 0$.

Ex 2.

Explain how to define $u \in H^1(\Omega)$ to be a weak solution of Poisson's equation with *Robin boundary conditions*:

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u + \frac{\partial u}{\partial \eta} &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{4}$$

Discuss the existence and uniqueness of a weak solution for a given $f \in L^2(\Omega)$.

Ex 3.

Assume Ω is connected and $f \in L^2(\Omega)$. Suppose $\partial\Omega$ is *cut* into two disjoints, closed sets Γ_1 and Γ_2 (see Fig. 1). Define what it means for u to be a weak solution of Poisson's equation with *mixed Dirichlet-Neumann's boundary conditions*:

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma_1 \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{on } \Gamma_2 \end{cases} \tag{5}$$

Discuss the existence and uniqueness of a weak solution.

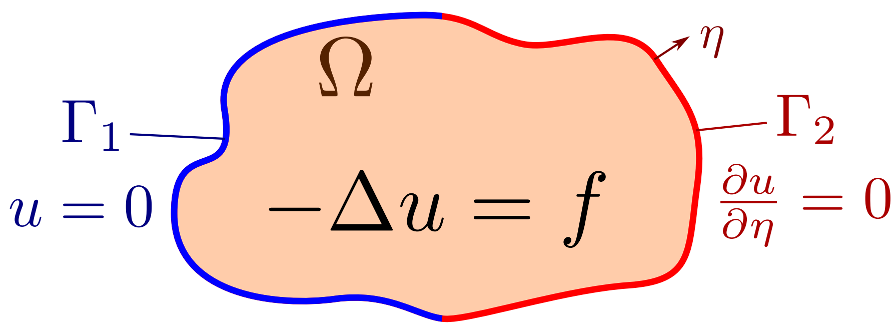

Figure 1: Mixed Dirichlet-Neumann's boundary conditions (5): $u = 0$ on a piece Γ_1 of the boundary $\partial\Omega$ and $\frac{\partial u}{\partial \eta} = 0$ on the other part Γ_2.

2 Maximum principle

Ex 4.

Let u be a smooth solution of the uniformly elliptic equation:

$$Lu := -\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i x_j} = 0 \qquad \text{in } \Omega.$$

Assume that the coefficients a^{ij} have bounded derivatives (i.e. $a^i \in W^{1,\infty}(\Omega)$).

a) Set $v := |\nabla u|^2 + \lambda u^2$ and show that for large enough λ:

$$Lv \leq 0 \qquad \text{in } \Omega.$$

b) Deduce

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\Big(\|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\Big).$$

Ex 5.

Assume Ω is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem:

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{6}$$

are the constant functions $u = C$.

3 Eigenvalues

Ex 6.

Let $Lu = -\sum_{i,j=1}^{n}(a^{ij}(x)u_{x_i})_{x_j}$ uniformly elliptic operator (in particular a^{ij} symmetric) on the domain Ω with zero boundary condition (i.e. $u \in H_0^1(\Omega)$).
A scalar λ is called an eigenvalue of L if there exists a non-zero $\omega_k \in H_0^1(\Omega)$ such that:

$$L\omega_k = \lambda_k \omega_k.$$

a) Show that the eigenvalues λ_i are real and strictly positive.
We denote its spectrum as:

$$0 < \lambda_1 \leq \lambda_2 \leq \dots$$

b) Show that:

$$\lambda_1 = \min_{u \in H_0^1\,,\, \|u\|_{L^2}=1} B[u,u],$$

with $B[u,u] = \langle -Lu, u\rangle_{H^1} = \int_\Omega a^{ij}u_{x_i}u_{x_j}\,\mathrm{d}x$.

c) Show that:

$$\lambda_2 = \min_{u \perp \omega_1\,,\, \|u\|_{L^2}=1} B[u,u],$$

and more generally

$$\lambda_k = \min_{u \perp \mathrm{Span}(\omega_1,\dots,\omega_{k-1})\,,\, \|u\|_{L^2}=1} B[u,u]. \tag{7}$$

Remark. The formula (7) can be written in a more "obscure way" as a 'max-min' formula:

$$\lambda_k = \max_{S \in \sum_{k-1}} \min_{u \in S^\perp\,,\, \|u\|_{L^2}=1} B[u,u]. \tag{8}$$

where $\sum_{k-1}$ is the collection of all $k-1$ dimensional subspaces of $H_0^1(\Omega)$. The 'max' is actually when $S = \mathrm{Span}(\omega_1, \dots, \omega_{k-1})$.