APM 576: Homework 4 (11/13)

1 Galerkin approximation

Ex 1.

Suppose $f \in L^2(\Omega)$. Denote $V_N = \text{Span}(\omega_1, \ldots, \omega_N)$ with ω_k eigenfunction of the operator $\mathcal{L}u = -\Delta u$ on $H_0^1(\Omega)$. Suppose $u_N \in V_N$ (i.e. $u_N = \sum_{k=1}^N d_k \omega_k$) solution to:

$$\int_\Omega \nabla u_N \cdot \nabla \omega_k \, \mathrm{d}x = \int_\Omega f \omega_k \, \mathrm{d}x \quad , \qquad \text{for } k = 1 \ldots N.$$

a) Show that a subsequence of $\{u_N\}_N$ converges weakly in $H_0^1(\Omega)$ to the weak solution u of

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

Hint: use the Poincaré inequality to show that the sequence $\{u_N\}_N$ is (uniformly) bounded in H^1.

b) Suppose that another subsequence of $\{u_N\}_N$ converges in $H_0^1(\Omega)$ to a limit v. Show that $v = u$ (i.e. uniqueness of the limit point).

c) Deduce that *all* the sequence $\{u_N\}_N$ converges weakly to u in $H_0^1(\Omega)$.

Hint: proceed by contradiction.

Remark. One can show that the sequence $\{u_N\}_N$ converges *strongly* in $L^2(\Omega)$ using Pythagorean theorem (i.e. Parseval's identity in an Hilbert space).

Ex 2.

a) Consider again the family of eigenfunctions $\{\omega_k\}_k$ from **Ex 1.**
Show that $\{\omega_k\}_k$ is also an orthogonal basis in $H_0^1(\Omega)$, i.e. $\omega_i \perp \omega_j$ in $H_0^1(\Omega)$ for $i \neq j$:

$$\int_\Omega \omega_i \omega_j \, \mathrm{d}x + \int_\Omega \nabla \omega_i \cdot \nabla \omega_j \, \mathrm{d}x = 0 \qquad \text{if } i \neq j,$$

and $\{\omega_k\}_k$ complete in $H_0^1(\Omega)$.

b) Consider now a general self-adjoint operator $\tilde{\mathcal{L}}$, i.e. $a^{ij} = a^{ij}$ unif. elliptic, $b^i = 0$ and $c \geq 0$. Let $\{\tilde{\omega}_k\}_k$ be the eigenfunctions of $\tilde{\mathcal{L}}$.
Explain why the family $\{\tilde{\omega}_k\}_k$ is not necessarily orthogonal in $H_0^1(\Omega)$ (even though it is always orthogonal in $L^2(\Omega)$).

2 Linear evolution eq. (smooth solution)

Suppose Ω open, bounded set of $\mathbb{R}^n$ with smooth boundary and $T > 0$ a fixed time.

Ex 3.

Prove there is at most one smooth solution to the problem:

$$\left\{\begin{array}{rcll} \partial_t \mathbf{u} - \Delta \mathbf{u} & = & f & \text{in } \Omega \times (0,T) \\ \frac{\partial \mathbf{u}}{\partial \eta} & = & 0 & \text{on } \partial\Omega \times [0,T] \\ \mathbf{u} & = & g & \text{on } \Omega \times \{t=0\}. \end{array}\right. \tag{2}$$

Ex 4.

Assume $\mathbf{u}$ smooth solution to

$$\left\{\begin{array}{rcll} \partial_t \mathbf{u} - \Delta \mathbf{u} & = & 0 & \text{in } \Omega \times (0,T) \\ \mathbf{u} & = & 0 & \text{on } \partial\Omega \times [0,T] \\ \mathbf{u} & = & g & \text{on } \Omega \times \{t=0\}. \end{array}\right. \tag{3}$$

Prove the exponential decay estimate for $t \geq 0$:

$$\|\mathbf{u}(\cdot,t)\|_{L^2} \leq \mathrm{e}^{-\lambda_1 t} \|g\|_{L^2}$$

where λ_1 is the principal eigenvalue of $-\Delta$ on Ω (with Dirichlet boundary conditions).

3 Linear evolution eq. (construction solution)

Ex 5.

Consider $V_N = \mathrm{Span}(\omega_1, \ldots, \omega_N)$ with ω_i defined in **Ex 1.** Denote $\mathbf{u}_N$ the (weak) solution to the heat equation on V_N, i.e. for any $v \in V_N$

$$\langle \mathbf{u}_N'(t), v \rangle + \langle \mathcal{L}\mathbf{u}_N(t), v \rangle \; = \; \langle \mathbf{f}(t), v \rangle,$$

for (almost every) $0 \leq t \leq T$ where $\langle \, , \rangle$ is the usual inner product in $L^2(\Omega)$.

Assume:

$$\left\{\begin{array}{ll} \mathbf{u}_N \rightharpoonup \mathbf{u} & \text{in } L^2(0,T; H_0^1(\Omega)) \\ \mathbf{u}_N' \rightharpoonup \mathbf{v} & \text{in } L^2(0,T; H_0^{-1}(\Omega)). \end{array}\right.$$

Prove that $\mathbf{v} = \mathbf{u}'$.

Hint: let $\phi \in C_c^1(0,T)$ and $\omega \in H_0^1(\Omega)$ then

$$\int_0^T \langle \mathbf{u}_N', \, \phi\omega \rangle \, \mathrm{d}t = -\int_0^T \langle \mathbf{u}_N, \, \phi\omega \rangle \, \mathrm{d}t.$$

Ex 6.

Assume as in **Ex 5.** that: $\mathbf{u}_N \rightharpoonup \mathbf{u}$ in $L^2(0,T;H_0^1(\Omega))$.
Suppose moreover that for any $0 \le t \le T$, $\mathbf{u}_N(t)$ is (uniformly) bounded in $H_0^1(\Omega)$, i.e.

$$\operatorname*{ess\,sup}_{0\le t\le T} \|\mathbf{u}_N(t)\|_{H^1} \le C$$

for all N. Deduce that:

$$\operatorname*{ess\,sup}_{0\le t\le T} \|\mathbf{u}(t)\|_{H^1} \le C,$$

i.e. $\mathbf{u} \in L^\infty(0,T;H_0^1(\Omega))$.