MAT 475: Solution homework 5 (10/4)

1 Chapter 6

Ex 3. [2pts]

We take a rotation matrix along the z-axis:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Solutions are given as:

$$\mathbf{x}(t) = \begin{pmatrix} x_0 \cos t - y_0 \sin t \\ x_0 \sin t + y_0 \cos t \\ z_0 \end{pmatrix}.$$

If $x_0 = y_0 = 0$, then the solution is constant. Otherwise, the solution is periodic with period 2π.

Ex 5. [3pts]

The characteristic polynomial is given by $p_A(\lambda) = (\lambda + a)(\lambda - a)(b - \lambda)$. Thus, we find as eigenvectors: 1pt

$$\lambda = b: \ \mathbf{v}_1 = (0,1,0) \quad , \quad \lambda = a: \ \mathbf{v}_2 = (1,0,1) \quad , \quad \lambda = -a: \ \mathbf{v}_3 = (1,0,-1).$$ 1pt

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{bt}\mathbf{v}_1 + c_2 \mathrm{e}^{at}\mathbf{v}_2 + c_3 \mathrm{e}^{-at}\mathbf{v}_3.$$ 1pt

Ex 6.

The characteristic polynomial is given by $p_A(\lambda) = (\lambda^2 - 2a\lambda + a^2 + b^2)(b - \lambda)$. Thus, we find as eigenvectors:

$$\lambda = b: \ \mathbf{v}_1 = (0,1,0) \quad , \quad \lambda = a + ib: \ \mathbf{w}_2 = (1,0,i) = \mathbf{u}_2 + i\mathbf{v}_2.$$

General solutions:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{bt}\mathbf{v}_1 + \mathrm{e}^{at}\Big(c_2(\cos bt \cdot \mathbf{u}_2 - \sin bt \cdot \mathbf{v}_2) + c_3(\sin bt \cdot \mathbf{u}_2 + \cos bt \cdot \mathbf{v}_2)\Big).$$

Ex 12. [5pts] (10 × .5)

a) $A = PDP^{-1}$ with $D = diag(-1, 2)$ and $P = [\mathbf{v}_1\mathbf{v}_2]$ with $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (2, 1)$. Thus,

$$e^A = P \begin{bmatrix} e^{-1} & 0 \\ 0 & e^2 \end{bmatrix} P^{-1}.$$

b) $e^A = e^2 \begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix}$.

c) $e^A = e^2 \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

d) $A = PDP^{-1}$ with $D = diag(-1, 1)$ and $P = [\mathbf{v}_1\mathbf{v}_2]$ with $\mathbf{v}_1 = (1, -1)$ and $\mathbf{v}_2 = (1, 1)$. Thus,

$$e^A = P \begin{bmatrix} e^{-1} & 0 \\ 0 & e^1 \end{bmatrix} P^{-1}.$$

e) Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Then $e^A = \mathrm{Id} + A + \frac{A^2}{2}$, since $A^3 = 0$.

f) $e^A = \begin{bmatrix} e^2 & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & e^3 & e^3 \end{bmatrix}$.

g) $e^A = e^\lambda \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix}$.

h) $e^A = \begin{bmatrix} e^i & 0 \\ 0 & e^{-i} \end{bmatrix}$.

i) $e^A = e^{1+i} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

j) Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Notice that $A^2 = A$. Thus, $A^n = A$ for any n. Therefore,

$$\begin{aligned} e^A &= \mathrm{Id} + A + A^2/2 + A^3/3! + \ldots \\ &= \mathrm{Id} + A\Big(0 + 1 + 1/2 + 1/3! + \ldots\Big) = \begin{bmatrix} e^1 & 0 & 0 & 0 \\ e^1 - 1 & 1 & 0 & 0 \\ e^1 - 1 & 0 & 1 & 0 \\ e^1 - 1 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Ex 13.

Take $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Then:

$$AB = \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 1 & 4 \\ 3 & 8 \end{bmatrix}.$$

Thus, $AB \neq BA$.