MAT 475: Solution homework 6 (10/16)

1 Chapter 7

Ex 1. [3pts]

a) $x' = x + 2$, $x(0) = 2$. Picard's iteration: $x_0(t) = 2$. Then:

$$\begin{aligned}
x_1(t) &= 2 + \int_0^t f(x_0(s))\,\mathrm{d}s = 2 + \int_0^t (2+2)\,\mathrm{d}s = 2 + 4t, & \boxed{.5\text{pt}}\\
x_2(t) &= 2 + \int_0^t (2+2+4s)\,\mathrm{d}s = 2 + 4t + 2t^2,\\
x_3(t) &= 2 + \int_0^t (4+4s+2s^2)\,\mathrm{d}s = 2 + 4t + 2t^2 + 2t^3/3,\\
x_4(t) &= 2 + \int_0^t (4+4s+2s^2+2s^3/3)\,\mathrm{d}s = 2 + 4t + 2t^2 + 2t^3/3 + 2t^4/3\cdot 4\\
&\vdots\\
x_n(t) &= 2 + 4t + 4t^2/2 + \cdots + 4t^n/n! & \boxed{.5\text{pt}}
\end{aligned}$$

We recognize the Taylor series of the function $x(t) = 4\mathrm{e}^t - 2$. Thus, the sequence converges in n for any t.

.5pt

b) $x_n(t) = 0$ for all n.

c) $x' = x^{4/3}$, $x(0) = 1$. Picard's iteration: $x_0(t) = 1$. Then:

$$\begin{aligned}
x_1(t) &= 1 + \int_0^t 1\,\mathrm{d}s = 1 + t, & \boxed{.5\text{pt}}\\
x_2(t) &= 1 + \int_0^t (1+s)^{4/3}\,\mathrm{d}s = 1 + \frac{(1+t)^{7/3}}{3/7},\\
x_3(t) &= 1 + \int_0^t \left(1 + \frac{(1+s)^{7/3}}{3/7}\right)^{4/3}\,\mathrm{d}s = ??
\end{aligned}$$

The solution is given by:

$$x^{-4/3}\cdot x' = 1 \quad\Rightarrow\quad -3x^{-1/3}(t) + 3x^{-1/3}(0) = t \quad\Rightarrow\quad x(t) = \frac{1}{(1-t/3)^3}.$$

The solution is defined up to $t = 3$. We deduce that the Picard iterations cannot converge for $t \geq 3$.

d) $x' = \cos x$, $x(0) = 0$. Picard's iteration: $x_0(t) = 0$. Then:

$$\begin{aligned} x_1(t) &= 0 + \int_0^t \cos(0)\,\mathrm{d}s = t, \\ x_2(t) &= 0 + \int_0^t \cos s\,\mathrm{d}s = -\sin t, \\ x_3(t) &= 0 + \int_0^t \cos(-\sin s))\,\mathrm{d}s = ?? \end{aligned}$$

.5pt

There is a "trick" to find the explicit solution: denote $y(t) = \sin(x(t))$, Then:

$$y' = \cos x \cdot x' = \cos^2 x = 1 - \sin^2 x = 1 - y^2$$

We can solve $y' = 1 - y^2$ and then deduce that $x(t) = \arcsin(y(t))$. The solution is defined for all time.

e) $x' = 1/2x$, $x(1) = 1$. Picard's iteration: $x_0(t) = 1$. Then:

$$\begin{aligned} x_1(t) &= 1 + \int_{\mathbf{1}}^t f(x_0(s))\,\mathrm{d}s = 1 + \int_{\mathbf{1}}^t \frac{1}{2}\,\mathrm{d}s = \frac{1}{2} + \frac{t}{2}, \\ x_2(t) &= 1 + \int_{\mathbf{1}}^t \frac{1}{1+s}\,\mathrm{d}s = 1 + \ln(1+t) - \ln 2 \\ x_3(t) &= 1 + \int_{\mathbf{1}}^t \frac{1}{1 + \ln(1+t) - \ln 2}\,\mathrm{d}s = \ ?? \end{aligned}$$

.5pt

The solution to the differential equation is given by: $x(t) = \sqrt{t}$. We deduce that the Picard's iterations cannot converge for $|t - 1| \geq 1$.

Ex 2. [3pts]

We apply the Picard method to solve the linear system: $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$. We first have: $x_0(t) = x_0$. Then:

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{x}_0 + \int_0^t A\mathbf{x}_0\,\mathrm{d}s = \mathbf{x}_0 + tA\mathbf{x}_0, \\ \mathbf{x}_2(t) &= \mathbf{x}_0 + \int_0^t (A\mathbf{x}_0 + sA^2\mathbf{x}_0)\,\mathrm{d}s = \mathbf{x}_0 + tA\mathbf{x}_0 + t^2A^2\mathbf{x}_0/2, \\ \mathbf{x}_3(t) &= \mathbf{x}_0 + \int_0^t (A\mathbf{x}_0 + sA^2\mathbf{x}_0 + s^2A^3\mathbf{x}_0/2)\,\mathrm{d}s = \mathbf{x}_0 + tA\mathbf{x}_0 + t^2A^2\mathbf{x}_0/2 + t^3A^3\mathbf{x}_0/3!, \\ &\ \vdots \\ \mathbf{x}_n(t) &= \mathbf{x}_0 + tA\mathbf{x}_0 + t^2A^2\mathbf{x}_0/2 + \cdots + t^nA^n\mathbf{x}_0/n! \end{aligned}$$

1pt

1pt

We recognize the Taylor expansion of the exponential of e^{tA}. We deduce that: $\mathbf{x}_n(t) \overset{n\to\infty}{\longrightarrow} \mathrm{e}^{tA}\mathbf{x}_0$ solution to the linear-system. 1pt

Ex 3.

The solution to $x'' = -4x$ with $x(0) = 0$ and $x'(0) = 2$ is given by $x(t) = \sin 2t$. Writing the equation into a linear system leads to:

$$x' = Ax \quad \text{with} \quad A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \quad \text{and} \quad x(0) = (0, 2).$$

The Picard's iterations give:

$$\begin{aligned}
\mathbf{x}_1(t) &= \mathbf{x}_0 + \int_0^t A\mathbf{x}_0\,\mathrm{d}s = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \int_0^t \begin{pmatrix} 2 \\ 0 \end{pmatrix} \mathrm{d}s = \begin{pmatrix} 2t \\ 2 \end{pmatrix} = \begin{pmatrix} 2t \\ 2 \end{pmatrix} \\
\mathbf{x}_2(t) &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \int_0^t \begin{pmatrix} 2 \\ -8s \end{pmatrix} \mathrm{d}s = \begin{pmatrix} 2t \\ 2-4t^2 \end{pmatrix} \\
\mathbf{x}_3(t) &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} + \int_0^t \begin{pmatrix} 2-4s^2 \\ -8s \end{pmatrix} \mathrm{d}s = \begin{pmatrix} 2t-4t^3/3 \\ 2-4t^2 \end{pmatrix} = \begin{pmatrix} 2t-(2t)^3/3! \\ 2(1-(2t)^2/2!) \end{pmatrix}
\end{aligned}$$

We recognize the Taylor series of $\sin 2t$ and $2\cos 2t$.

Ex 4.

Suppose $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are two solutions to the linear, non-autonomous system: $\mathbf{x}' = A(t)\mathbf{x}$. Then for any scalars α and β:

$$(\alpha\mathbf{x} + \beta\mathbf{y})' = \alpha\mathbf{x}' + \beta\mathbf{y}' = \alpha A(t)\mathbf{x} + \beta A(t)\mathbf{y} = A(t)(\alpha\mathbf{x} + \beta\mathbf{y}).$$

Thus, $\alpha\mathbf{x} + \beta\mathbf{y}$ is also a solution to $\mathbf{x}' = A(t)\mathbf{x}$.

Ex 6. [4pts]

Let $a > 0$ and consider the differential equation: $x' = x^a$ with $x(0) = 0$. The zero function $x(t) = 0$ is always a solution. To study uniqueness, let's try to find another one. 1pt

First, if $a = 1$, then $x'/x = 1$ leads to: $x(t) = ke^t$. Using the initial condition, we deduce $x(t) = 0$. Thus, we obtain the same solution. .5pt

If $a \neq 1$, then

$$\frac{x'}{x^a} = 1 \quad \Rightarrow \quad \frac{x^{-a+1}}{-a+1} = t + c \quad \Rightarrow \quad x(t) = \Big((1-a)(t+c)\Big)^{\frac{1}{1-a}}.$$ .5pt

The initial condition leads to $c = 0$. Thus, $x(t) = \Big((1-a)t\Big)^{\frac{1}{1-a}}$. This solution is defined at $t = 0$ only if $\frac{1}{1-a} > 0$ meaning $0 < a < 1$. We conclude that if $0 < a < 1$ there is no uniqueness of solution to $x' = x^a$ with $x(0) = 0$. .5pt .5pt 1pt

Remark. When $0 < a < 1$, the function $f(x) = x^a$ is no longer differentiable at $x = 0$.