# MAT 475: Solution homework 7 (10/23)

## 1 Chapter 8

**Ex 1.** [4pts]

i) Equilibria: $(m\pi, n\pi + \pi/2)$.
Stability: $DF(x_*) = \begin{bmatrix} (-1)^m & 0 \\ 0 & (-1)^{n+1} \end{bmatrix}$. The equilibrium is a saddle if $(m, n)$ are both even or odd, a source if $m$ is even and $n$ odd, a sink if $m$ is odd and $n$ even. Since the eigenvalues are never zero, the linear system accurately describes the non-linear system near equilibrium. 1pt

ii) Equilibrium: $(0, 0)$. Jacobian is zero, the linear-system consists only of constant solution.
To understand the behavior of the non-linear system, we use polar coordinates $x = r\cos\theta,\ y = r\sin\theta$: 1pt

$$\begin{aligned} r' &= \frac{xx' + yy'}{r} = r^3 \\ \theta' &= \frac{-yx' + xy'}{r^2} = 0. \end{aligned}$$

We deduce that if $r(0) \neq 0$ the solution $r(t)$ is going to infinity in finite time. Thus, the linearized system does not describe the behavior of the non-linear system.

iii) Equilibrium: $(0, 0)$. Stability: $DF(x_*) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, The equilibrium is a source, the linear system accurately describes the behavior of the solutions near the origin. 1pt

iv) Equilibrium: $(x, 0)$. Stability: $DF(x_*) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, since there is a zero eigenvalue, we cannot conclude. The solutions of the linearized system are simply moving in the $y$ direction (away from the $y$-axis). For the non-linear system, the solutions are also moving in the $x-$direction. The linearized system failed (again) to reproduce locally the behavior of the non-linear system. 1pt

v) Equilibrium: $(0, 0)$. Jacobian is zero, the linear-system consists only of constant solutions. Whereas for the non-linear system, solutions are either approaching zeros (if $x_0 < 0$ and $y_0 < 0$) or moving away from the origin.

**Ex 2.**

We look for a change of variable of the form: $u = x + cy^2$. We find:

$$u' = x' + 2cyy' = x + y^2 - 2cy^2 = x + (1-2c)y^2$$

In order to have $u' = u$, we need to take $1 - 2c = c$ which leads to $c = 1/3$. Similar, if we look for $w = z + cy^2$, we find $c = 1$. Thus, the change of coordinates is given by:

$$u = x + \frac{y^2}{3}, \; v = y, \; w = z + y^2.$$

**Ex 5.** [3pts]

a) Equilibrium: $y = -x^2$ thus $x$ has to satisfy $x^2 + x + a = 0$. Therefore,

$$x_\pm = \frac{-1 \pm \sqrt{1-4a}}{2}.$$ 1pt

There is no equilibrium if $a > 1/4$, one equilibrium if $a = 1/4$ and two equilibria if $a < 1/4$.

The linearized system is given by:

$$A = DF(x_*) = \begin{bmatrix} 2a & 1 \\ 1 & -1 \end{bmatrix}$$

b) We suppose $a \leq 1/4$. At the equilibrium point, we have:

$$\det(A) = -2a - 1 = \mp\sqrt{1-4a}.$$

Thus, at $x_1 = \frac{-1-\sqrt{1-4a}}{2}$, the determinant is positive, and the trace is negative, thus the equilibrium is a sink. At $x_2 = \frac{-1-\sqrt{1-4a}}{2}$, the determinant is negative, therefore the equilibrium is a saddle. 1pt

c) At $a = 1/4$, the system goes from one equilibrium to two. We therefore have a saddle-node bifurcation. 1pt

**Ex 6.**

To have 4 equilibria when $a$ is positive, we take for the function $f$ a polynomial of order 4 with roots at $\pm\sqrt{a}$ and $\pm\sqrt{2a}$:

$$f(x) = (x^2 + a)(x^2 + 2a).$$

The bifurcation diagram is given in figure 1.

**Ex 7.** [3pts]

The solution of $r' = r - r^3$ has two equilibria (since $r \geq 0$) at $r = 0$ (unstable) and at $r = 1$ (stable). Moreover, the line space indicates that if $r(0) \neq 0$, we necessary have: $r(t) \overset{t\to+\infty}{\longrightarrow} 1$. 1pt



1+1pt

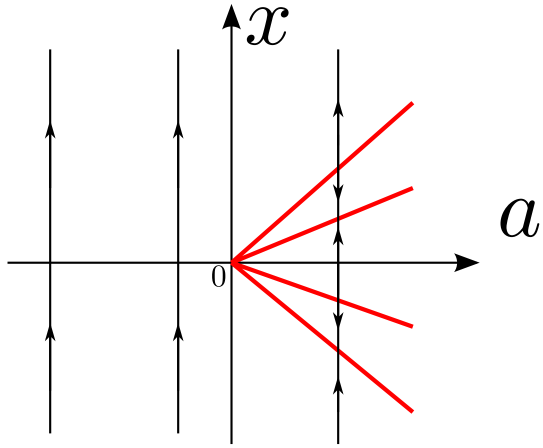


Figure 1: Bifurcation diagram for the function $f(x) = (x^2 + a)(x^2 + 2a)$.

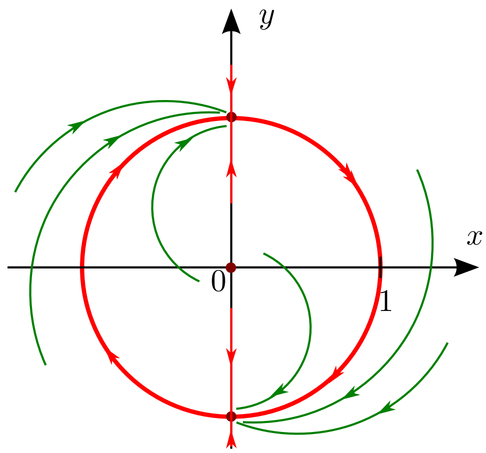


Figure 2: Phase portrait for the system: $r' = r - r^3$, $\theta' = \sin^2\theta - 1$.

For the equation $\theta' = \sin^2\theta - 1$, the solution is $\theta(t)$ is always decreasing (i.e. the solution turns clock-wise) with two (unstable) equilibria at $\pm\pi/2$. We deduce the phase portrait given in figure 2.

**Ex 12.**

Consider the following system in polar coordinates:

$$\begin{aligned} r' &= r(1-r) \\ \theta' &= \sin^2(\theta/2) \end{aligned}$$

The point $(r, \theta) = (1, 0)$ (i.e. $x = 1$, $y = 0$) is an attractor: all the trajectories (except the zero solution) converge toward this point. But this equilibrium is not stable. You can check the phase portrait with the web solver.