**MAT 475: Midterm 1 Sept. 25, 2017 Duration: 1:15**

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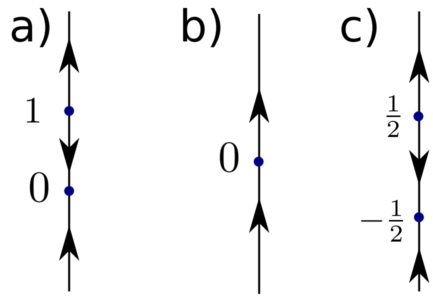
**Exercise 1.** [25pts]

a) $f(x) = x^2 - x$. Equilibria ($f(x) = 0$): $x = 0$ unstable ($f'(0) < 0$), $x = 1$ stable ($f'(1) > 0$). 5pts

b) $f(x) = \sqrt{|x|}$. Equilibrium: $x = 0$ unstable. 5pts

c) $f(x) = \ln(3/4 + x^2)$. Equilibria: $x = -1/2$ sink, $x = 1/2$ source. 6pts

The phase lines are given in figure 1.




3+3+3pts

Figure 1: Phase lines for exercise 1.

**Exercise 2.** [30pts]

a) For $A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$. Eigenvalues/eigenvectors:

$$\lambda_1 = -2, \quad \mathbf{v}_1 = (1, -1) \quad , \quad \lambda_2 = 1, \quad \mathbf{v}_2 = (2, 1).$$ 3pts

General solutions:

$$\mathbf{x}(t) = c_1 e^{-2t}\mathbf{v}_1 + c_2 e^{t}\mathbf{v}_2.$$ 3pts

For $B = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$. Eigenvalues/eigenvectors:

$$\lambda = 2 \pm i, \; \mathbf{w} = (2, 1 + i), = \mathbf{u} + i\mathbf{v},$$ 2pts

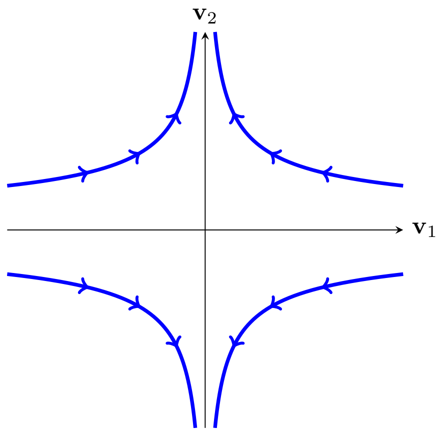
with $\mathbf{u} = (2, 1)^T$, $\mathbf{v} = (0, 1)^T$. 2pts

General solutions:

$$\mathbf{x}(t) = c_1 e^{2t}(\cos t\,\mathbf{u} - \sin t\,\mathbf{v}) \; + \; c_2 e^{2t}(\sin t\,\mathbf{u} + \cos t\,\mathbf{v}).$$ 4pts

b) For $A$, take $T = [\mathbf{v}_1 \; \mathbf{v}_2] = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$. Then: $T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$. 2+2pts

For $B$, take $T = [\mathbf{u} \; \mathbf{v}] = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$. Then: $T^{-1}BT = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. 2+2pts



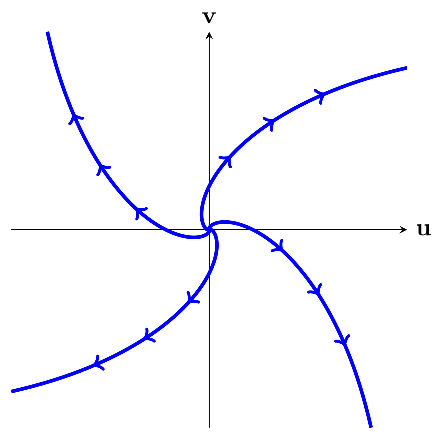




Figure 2: Phase portrait for Exercise 2: **left** figure for $A$ (saddle point) and **right** figure for $B$ (complex eigenvalues).

c) The phase portrait are given in figure 2. 4+4pts

**Exercise 3.** [25pts]

a) $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}$. 6pts

b) In order to have oscillating solution, we need to have complex eigenvalues. Thus, we need to have: $T^2 - 4D < 0 \Rightarrow \underline{b^2 - 4k < 0}$. 5pts

To obtain a center, we can take $b = 0$, $\kappa = 1$ (trace is zero). To have a focus, we can take $b = 1$, $\kappa = 1$ (the condition $b^2 - 4\kappa < 0$ is satisfied). 3+3pts

c) The solution satisfies $x(2\pi) = 1$ if it is a periodic solution. Thus, we need to have center and therefore $b = 0$. The periodicity of the solution is given by: $\beta = \sqrt{|b^2 - 4\kappa|}/2 = \kappa$. We take $\kappa = 1$ to have periodicity $2\pi$. 8pts

*Analytic version*: the general solution (if $b^2 - 4\kappa < 0$) is given by:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{\alpha t}(\cos\beta t\ \mathbf{u} - \sin\beta t\ \mathbf{v}) + c_2 \mathrm{e}^{\alpha t}(\sin\beta t\ \mathbf{u} + \cos\beta t\ \mathbf{v})$$

with $\alpha = -b/2$ and $\beta = \sqrt{|b^2 - 4\kappa|}/2$. If $b = 0$ and $\kappa = 1$, then $\alpha = 0$ and $\beta = 1$:

$$\mathbf{x}(t) = c_1(\cos t\ \mathbf{u} - \sin t\ \mathbf{v}) + c_2(\sin t\ \mathbf{u} + \cos t\ \mathbf{v}).$$

We deduce: $\mathbf{x}(2\pi) = \mathbf{x}(0)$.

Extra) We still require $\beta = 1$, thus $4\kappa - b^2 = 4$. Then, 5pts

$$\mathbf{x}(2\pi) = \mathrm{e}^{\alpha 2\pi}\mathbf{x}(0).$$

To have $x(2\pi) = 1/2$, we need $e^{\alpha 2\pi} = 1/2$:

$$e^{-b\pi} = \frac{1}{2} \quad \Rightarrow \quad -b\pi = -\ln 2 \quad \Rightarrow \quad b = \frac{\ln 2}{\pi}.$$

Then, using $4\kappa - b^2 = 4$, we deduce: $k = 1 + b^2/4 = 1 + \left(\frac{\ln 2}{2\pi}\right)^2$.

**Exercise 4.** [20pts]

- The characteristic polynomial of $A$ is given (developing along the 3rd row):

$$P_A(\lambda) = (1-\lambda)(\lambda^2 + \lambda - 6) = (1-\lambda)(\lambda - 2)(\lambda + 3).$$ 4pts

The eigenvalues are distinct and real: $\lambda = -3, 1, 2$. Thus, the canonical form of $A$ is given by:

$$D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$ 4pts

- The characteristic of $B$ (using the 2nd row) is:

$$P_B(\lambda) = (\lambda + 1)(\lambda^2 - 0 \cdot \lambda + 0) = (\lambda + 1)\lambda^2.$$ 4pts

Thus, its eigenvalues are $\lambda = -1$ (single) and $\lambda = 0$ (repeated).

We have to determined whether there exist one or two eigenvectors for $\lambda = 0$:

$$(B - 0 \cdot \mathrm{Id})\mathbf{u} = \mathbf{0}.$$

Since two columns of $B$ are linearly independent, there exists only one solution (up to multiplication by a constant) to the previous equation. Thus, the second vector for $\lambda = 0$ will be a generalized eigenvectors. Therefore, the canonical form for $B$ is given by: 4pts

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 4pts