MAT 475: Practice midterm 2

Exercise 1. Describe the regions in a, b, c space where the matrix

$$A = \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ c & 0 & a \end{bmatrix}$$

has real, complex and repeated eigenvalues.

Exercise 2. Find the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ for the following matrices:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 2 \end{bmatrix}.$$

Exercise 3. We would like to find the solution to:

$$\begin{cases} x' &= x - y + \mathrm{e}^{-t} \\ y' &= -x + y \end{cases} \tag{1}$$

a) Find the exponential of the matrix tA with $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

b) Deduce the solution to (1).

Exercise 4. Consider the following differential equation:

$$\begin{cases} x' &= -x + t \\ x(0) &= 0 \end{cases}$$

a) Compute the three first iteration of the Picard method.

b) Find the expression of the n-th iteration and deduce the limit $n \to \infty$.

Exercise 5. Consider the non-linear system:

$$\begin{cases} x' &= 4x - 8 \\ y' &= x^2 - 4y^2 \end{cases}$$

a) Find the equilibria.

b) Characterize each equilibrium (e.g. sink, source, saddle)

c) Sketch the phase portrait near each equilibrium.

Exercise 6. Consider the non-linear system:

$$\begin{cases} x' &= x + \sin(3x - y) \\ y' &= e^x - 1 \end{cases}$$

a) Find the equilibria.

b) Characterize each equilibrium (e.g. sink, source, saddle)

Exercise 7. Consider the non-linear system:

$$\begin{cases} x' &= x^2 + y \\ y' &= x - y + a \end{cases}$$

where a is a parameter.

a) Find all the equilibria and compute the linearized equation at each.

b) Describe the behavior of the linearized system around each equilibrium.

c) Describe any bifurcations that occur.

Exercise 8.
Consider the dynamical system:

$$\begin{cases} x' &= x - x^3 \\ y' &= -y. \end{cases}$$

a) Find and study the stability of the equilibria.

b) Find and draw the nullclines.

c) Sketch the phase portrait.
What is the behavior of the solutions starting at $(x_0, 1)$ with $|x_0| < 1$?