APM 576: Homework 3 (10/23)

1 Boundary conditions

Ex 1.

a) Taking $v = 1 \in H^1(\Omega)$ in the weak formulation, we obtain:

$$0 = \int_\Omega f \, \mathrm{d}x.$$

Thus it is necessary to have such condition for f in order for u to be a weak-solution.

b) The key is to show the coercivity of the bilinear application $B[u, v]$. Applying Poincaré inequality, we have:

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)},$$

where $\bar{u}$ is the average of u on Ω and C is some positive constant which depends only on Ω. If we restrict the domain of functions to E, we have $\bar{u} = 0$ and thus:

$$\begin{aligned} B[u, u] &= \int_\Omega |\nabla u|^2 \, \mathrm{d}x = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2 \\ &\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2C}\|u\|_{L^2}^2 \\ &\geq \alpha \|u\|_{H^1}^2, \end{aligned}$$

with $\alpha = \min(1/2, 1/(2C))$. Thus B is coercive. Since B is also bounded (i.e. continuous), we can apply Lax-Milgram theorem to deduce that there exists a unique weal solution u on E.

The solution is not unique in $H^1(\Omega)$ since we can simply add a constant to u.

Ex 2.

• We first define the weak formulation of the probem. Using φ test function and assuming u smooth solution:

$$\begin{aligned} -\int_\Omega \Delta u \varphi \, \mathrm{d}x &= \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \varphi \, \mathrm{d}S \\ &= \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x + \int_{\partial\Omega} u \varphi \, \mathrm{d}S, \end{aligned}$$

using that $\frac{\partial u}{\partial \eta} = -u$ on $\partial\Omega$. Thus, the weak formulation is to find $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} uv \,\mathrm{d}S = \int_\Omega fv \,\mathrm{d}x \qquad \text{for all } v \in H^1(\Omega). \tag{1}$$

To show existence and uniqueness of a solution u, we have to show that the bilinear operator $B[u,v]$ defined as the left-hand side of equation (1) is coercive, i.e. there exists $\alpha > 0$ such that:

$$B[u,u] = \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} |u|^2 \,\mathrm{d}S \quad \geq \alpha \|u\|_{H^1}^2,$$

for any $u \in H^1(\Omega)$. This problem reduces to find a Poincaré constant C such that:

$$\int_\Omega |u|^2 \,\mathrm{d}x \leq C \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} |u|^2 \,\mathrm{d}S \right).$$

- We proceed by contradiction: suppose $u_n \in H^1(\Omega)$ such that

$$\int_\Omega |u_n|^2 \,\mathrm{d}x > n \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} |u|^2 \,\mathrm{d}S \right).$$

We normalize the sequence in $L^2(\Omega)$: $\omega_n = \frac{u_n}{\|u_n\|_{L^2}}$. The sequence $\{\omega_n\}_n$ satisfies $\|\omega_n\|_{L^2} = 1$ for all n and

$$\frac{1}{n} > \int_\Omega |\nabla \omega_n|^2 \,\mathrm{d}x + \int_{\partial\Omega} |\omega_n|^2 \,\mathrm{d}S. \tag{2}$$

In particular, $\|\omega_n\|_{H^1} \leq 2$. By Rellich' theorem, we can extract a subsequence that will converge in $L^2(\Omega)$: $\omega_{n_k} \overset{k\to+\infty}{\longrightarrow} \omega$ in $L^2(\Omega)$. The limit function ω satisfies $\|\omega\|_{L^2} = 1$ and $\nabla\omega = 0$. Indeed, taking $\varphi \in C_c^\infty(\Omega)$ test function:

$$\begin{aligned}
\int_\Omega \omega \partial_{x_i} \varphi \,\mathrm{d}x &= \lim_{n_k \to +\infty} \int_\Omega \omega_{n_k} \partial_{x_i} \varphi \,\mathrm{d}x \\
&= \lim_{n_k \to +\infty} \int_\Omega -\partial_{x_i} \omega_{n_k} \varphi \,\mathrm{d}x \\
&= 0,
\end{aligned}$$

since $\nabla\omega_n \to 0$. Thus $\partial_{x_i}\omega = 0$ for any x_i and therefore $\nabla\omega = 0$ (in particular $\omega \in H^1(\Omega)$). We deduce (see previous homework) that $\omega = C$ constant on Ω (since Ω connected) and thus its trace is $\Gamma(\omega) = C$ on $\partial\Omega$. To conclude, we need to show $C = 0$ and we will have a contradiction with $\|\omega\|_{L^2} = 1$.

We show that ω_{n_k} converges also in H^1 to ω. Indeed:

$$\begin{aligned}
\|\omega_{n_k} - \omega\|_{H^1}^2 &= \|\omega_{n_k} - \omega\|_{L^2}^2 + \|\nabla\omega_{n_k} - 0\|_{L^2}^2 \\
&\leq \|\omega_{n_k} - \omega\|_{L^2}^2 + \frac{1}{n_k} \quad \overset{n_k\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore, by continuity of the trace operator, $\Gamma(\omega_{n_k}) \overset{n_k\to+\infty}{\longrightarrow} \Gamma(\omega)$. Since $\Gamma(\omega_{n_k}) \to 0$ using (2), we deduce $\Gamma(\omega) = 0$. Since ω constant on Ω, we have $\omega = 0$ on Ω. Contradiction since $\|\omega\|_{L^2} = 1$.

• We can now conclude about the existence and uniqueness of a weak solution. The operator $B[u,v]$ is continuous and coercive:

$$\begin{aligned} B[u,u] &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x \quad + \quad \frac{1}{2}\left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \int_{\partial\Omega} |u|^2 \,\mathrm{d}S\right) \\ &\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 \quad + \quad \frac{1}{2C}\|u\|_{L^2}^2 \geq \alpha \|u\|_{H^1}^2. \end{aligned}$$

Thus, Lax-Milgram, it exists a unique weak solution u in $H^1(\Omega)$.

Ex 3.

Assuming u smooth solution and φ test function satisfying $\varphi = 0$ on Γ_1. We have:

$$\begin{aligned} -\int_\Omega \Delta u \varphi \,\mathrm{d}x &= \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \varphi \,\mathrm{d}S \\ &= \int_\Omega \nabla u \cdot \nabla \varphi \,\mathrm{d}x + 0, \end{aligned}$$

using $\frac{\partial u}{\partial \eta} = 0$ on Γ_2 and $\varphi = 0$ on Γ_1. Thus, denoting E the functional space,

$$E = \{u \in H^1(\Omega) \mid u|_{\Gamma_1} = 0\},$$

the weak formulation is to find $u \in E$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \qquad \text{for all } v \in E.$$

By continuity of the trace operator, the space E is an Hilbert space. Thus, to show the existence of weak-solution, it all boilds down to show the coercivity of the bilinear from $B[u,v] = \int_\Omega \nabla u \cdot \nabla v$ on E.

With this aim, we proceed by contradiction as in **Ex 2** and conclude that there exists ω in $L^2(\Omega)$ such that $\|\omega\|_{L^2} = 1$ and $\nabla \omega = 0$ (weakly). Thus, ω constant on Ω (since Ω connected). As $\omega_{n_k} \xrightarrow{H^1} \omega$, we also have $\omega|_{\Gamma_1} = 0$ (by continuity of the trace operator and using that $\omega_{n_k}|_{\Gamma_1} = 0$). Thus, $\omega = 0$. Contradiction. We can then conclude about the existence and uniqueness of a weak solution u in E.

2 Maximum principle

Ex 4.

a) Let $v := |\nabla u|^2 + \lambda u^2$. We estimate:

$$\begin{aligned} L(u^2) &= -2\sum_{ij} a^{ij} \partial_{x_i}(u\, u_{x_j}) = -2\sum_{ij} a^{ij}\left(u_{x_i} u_{x_j} + u\, u_{x_i x_j}\right) \\ &= -2\sum_{ij} a^{ij} u_{x_i} u_{x_j} + 0 \\ &\leq -2\theta |\nabla u|^2, \end{aligned}$$

using first $Lu = 0$ and then the uniform ellipticity of L. Moreover:

$$\begin{aligned} L(u_{x_k}^2) &= -2\sum_{ij} a^{ij}\partial_{x_i}(u_{x_k}\, u_{x_j x_k}) = -2\sum_{ij} a^{ij}\Big(u_{x_i x_k} u_{x_j x_k} + u_{x_k} u_{x_i x_j x_k}\Big) \\ &\leq -2\theta \sum_i |u_{x_i x_k}|^2 + u_{x_k}\sum_{ij} a^{ij} u_{x_i x_j x_k} \end{aligned}$$

using the uniform ellipticity. Moreover, $\partial_{x_k} L(u) = 0$, therefore:

$$-\sum_{ij}\partial_{x_k}(a^{ij}) u_{x_i x_j} = \sum_{ij} a^{ij} u_{x_i x_j x_k}.$$

Thus,

$$\begin{aligned} L(u_{x_k}^2) &\leq -2\theta\sum_i |u_{x_i x_k}|^2 - u_{x_k}\sum_{ij}\partial_{x_k}(a^{ij}) u_{x_i x_j} \\ &\leq -2\theta\sum_i |u_{x_i x_k}|^2 + C|u_{x_k}|\sum_{ij}|u_{x_i x_j}|, \end{aligned}$$

using the boundedness of $\partial_{x_k} a^{ij}$. We deduce:

$$L(|\nabla u|^2) = -2\theta|D^2 u|^2 + C\sum_k |u_{x_k}|\sum_{ij}|u_{x_i x_j}|.$$

Finally, we obtain:

$$L(v) \leq -2\lambda\theta|\nabla u|^2 - 2\theta|D^2 u|^2 + C\sum_k |u_{x_k}|\sum_{ij}|u_{x_i x_j}|. \tag{3}$$

We recognize an expression of the form: $-\lambda a^2 - b^2 + Cab$. To prove that this expression is negative for λ large enough, we use the following *trick*: $ab = \frac{a}{\sqrt{\varepsilon}} \cdot b\sqrt{\varepsilon} \leq \frac{1}{2}(\frac{a^2}{\varepsilon} + \varepsilon b^2)$ which leads here to:

$$\begin{aligned} C\sum_k |u_{x_k}|\sum_{ij}|u_{x_i x_j}| &\leq \frac{C}{2}\left(\frac{1}{\varepsilon}\Big(\sum_k |u_{x_k}|\Big)^2 + \varepsilon\Big(\sum_{ij}|u_{x_i x_j}|\Big)^2\right) \\ &\leq \frac{n\cdot C}{2}\left(\frac{1}{\varepsilon}|\nabla u|^2 + \varepsilon|D^2 u|^2\right), \end{aligned}$$

using Cauchy-Scwarz inequality, i.e. $(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$. Thus,

$$L(v) \leq \left(-2\lambda\theta + \frac{n\cdot C}{2\varepsilon}\right)|\nabla u|^2 \quad + \quad \left(-2\theta + \frac{n\cdot C}{2}\varepsilon\right)|D^2 u|^2.$$

To finish, we take $\varepsilon > 0$ small enough such that $-2\theta + \frac{n\cdot C}{2}\varepsilon < 0$, then take $\lambda > 0$ large enough such that $-2\lambda\theta + \frac{n\cdot C}{2\varepsilon} < 0$.

b) We can apply the maximum principle to v:

$$\begin{aligned} \max_{\overline{\Omega}} |\nabla u|^2 &\leq \max_{\overline{\Omega}}\{|\nabla u|^2 + \lambda|u|^2\} \leq \max_{\partial\overline{\Omega}}\{|\nabla u|^2 + \lambda|u|^2\} \\ &\leq C\max_{\partial\overline{\Omega}}\{|\nabla u|^2 + |u|^2\} \leq C\max_{\partial\overline{\Omega}}\{(|\nabla u| + |u|)^2\}. \end{aligned}$$

Taking the square root leads to the result.

Ex 5.

Assume Ω is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem:

$$\begin{cases} -\Delta u &= 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \eta} &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{4}$$

are the constant functions $u = C$.