MAT 475: Midterm 2 Nov. 6, 2017 Duration: 1:15

Exercise 1. [25pts]

a) We compute the characteristic polynomial:

$$\begin{aligned} p_A(\lambda) &= \begin{vmatrix} 4-\lambda & 0 & -6 \\ 1 & 1-\lambda & -1 \\ 3 & 0 & -5-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 4-\lambda & -6 \\ 3 & -5-\lambda \end{vmatrix} \\ &= (1-\lambda)(\lambda^2-\lambda+2) = -(1-\lambda)^2(\lambda+2). \end{aligned}$$

6pts

Thus, the eigenvalues of A are 1 and -2.

b) We look for the eigenvectors of A:

$$\lambda = -2: \quad \begin{bmatrix} 6 & 0 & -6 \\ 1 & 3 & -1 \\ 3 & 0 & -3 \end{bmatrix} \mathbf{u} = 0 \Rightarrow \mathbf{u}_1 = (1,0,1)$$

3pts

$$\lambda = 1: \quad \begin{bmatrix} 3 & 0 & -6 \\ 1 & 0 & -1 \\ 3 & 0 & -6 \end{bmatrix} \mathbf{u} = 0 \Rightarrow \mathbf{u}_2 = (0,1,0).$$

3pts

There is no other eigenvectors for $\lambda = 1$. Thus, we look for generalized eigenvector solving:

$$(A - 1\cdot\text{Id})\mathbf{w} = \mathbf{u}_2 \quad \Rightarrow \quad \mathbf{w} = (2,0,1).$$

4pts

The general solution is then given by:

$$\mathbf{x}(t) = c_1 e^{-2t}\mathbf{u}_1 + c_2 e^t \mathbf{u}_2 + c_3 e^t(\mathbf{w} + t\mathbf{u}_2).$$

3pts

c) If c_2 or c_3 are not zero (i.e. $c_2 \neq 0$ or $c_3 \neq 0$), then the solution diverges as $t \to +\infty$:

2pts

$$|\mathbf{x}(t)| \overset{t\to+\infty}{\longrightarrow} +\infty.$$

Otherwise ($c_2 = 0$ and $c_3 = 0$), the solution converges to the origin:

4pts

$$\mathbf{x}(t) \overset{t\to+\infty}{\longrightarrow} 0.$$

Exercise 2. [25pts]

a) The eigenvalues/eigenvectors of A are given by:

$$\lambda_1 = 0, \quad \mathbf{u}_1 = (1,1), \quad \lambda_1 = 1, \quad \mathbf{u}_2 = (2,1).$$

1+1+1+1pts

Thus, we can write:

3+3pts

$$A = T\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} T^{-1} \quad \text{with} \quad T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

We deduce:

$$
\begin{aligned}
e^{tA} &= T\begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}T^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}\frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ e^t & -e^t \end{bmatrix} = \begin{bmatrix} -1+2e^t & 2-2e^t \\ -1+e^t & 2-e^t \end{bmatrix}.
\end{aligned}
$$

3pts

3pts

b) The solution of the non-autonomous equation is given by:

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 + e^{tA}\int_0^t e^{-sA}\begin{pmatrix} s \\ s \end{pmatrix} ds.$$

3pts

Noticing that $e^{-sA}\begin{pmatrix} s \\ s \end{pmatrix} = \begin{pmatrix} s \\ s \end{pmatrix}$, we deduce:

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0 + e^{tA}\int_0^t \begin{pmatrix} s \\ s \end{pmatrix} ds = e^{tA}\mathbf{x}_0 + e^{tA}\begin{pmatrix} t^2/2 \\ t^2/2 \end{pmatrix}.$$

3+3pts

Exercise 3. [25pts]

a) Equilibria: solving $x' = 0$ yields $y = x^2$, $y' = 0$ leads to $y = x$. We deduce $x^2 = x$, thus $x = 0$ or 1. We obtain two equilibria:

$$(0,0), \quad (1,1).$$

3pts

Stability: we compute the Jacobian of the system

$$DF(x) = \begin{bmatrix} -2x & 1 \\ 1 & -1 \end{bmatrix}.$$

3pts

We deduce that:

$$
\begin{aligned}
\mathbf{x}_* = (0,0) \qquad & DF(\mathbf{x}_*) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad \textit{saddle} \\
\mathbf{x}_* = (2,2) \qquad & DF(\mathbf{x}_*) = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad \textit{sink}
\end{aligned}
$$

3pts

3pts

b) We plot the nullclines onto figure 1.
We deduce that the phase portrait is given by two sets of solutions: they are either going to the sink $(1,1)$ or diverge with $x(t) \overset{t\to+\infty}{\longrightarrow} -\infty$.

4pts

5pts

c) Solutions are trapped in the region $0 < x < 1$ and $0 < y < x^2$. They are all converging toward $(1,1)$.

5pts

Exercise 4. [25pts]

a) In polar coordinates:

$$\begin{aligned} r' &= \frac{xx' + yy'}{r} = \frac{x^2(1-r^2) + y^2(1-r^2)}{r} = \frac{r^2(1-r^2)}{r} = r(1-r^2) \\ \theta' &= \frac{-yx' + xy'}{r^2} = \frac{xy^2 + x^3}{r^2} = \frac{x(x^2+y^2)}{r^2} = x = r\cos\theta. \end{aligned}$$

6pts

The equilibria are at $r = 0$ ($\mathbf{x}_* = (0,0)$) and at $r = 1$ with $\theta = \pm\pi/2$ ($\mathbf{x}_* = (0, \pm 1)$).

4pts

b) Along the unit circle, the solutions are moving toward $(0, 1)$. We deduce that $\mathbf{x}_* = (0, -1)$ is unstable. Moreover, along the y-axis, the solutions are moving toward $(0, 1)$ and $(0, -1)$. Thus, $(0, -1)$ is a saddle equilibrium whereas $(0, 1)$ is a sink.

3+3+4pts

c) The phase portrait is given in fig. 2.

5pts

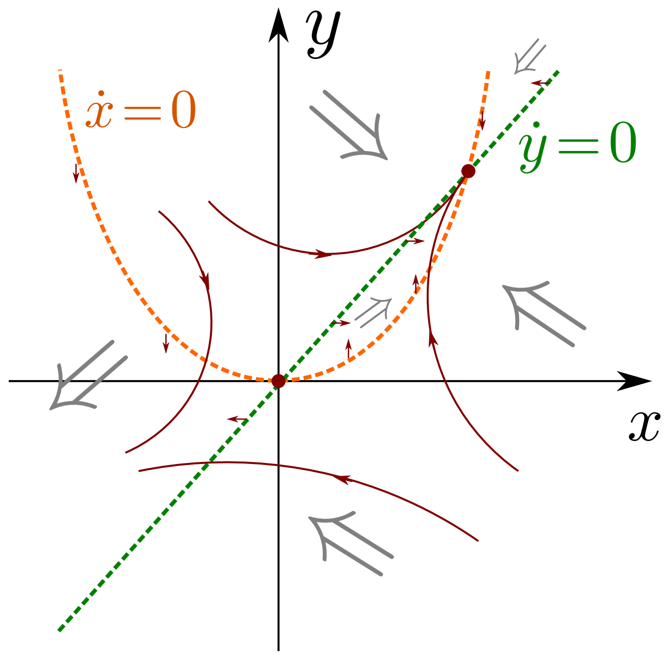

Figure 1: Phase portrait for exercise 3.

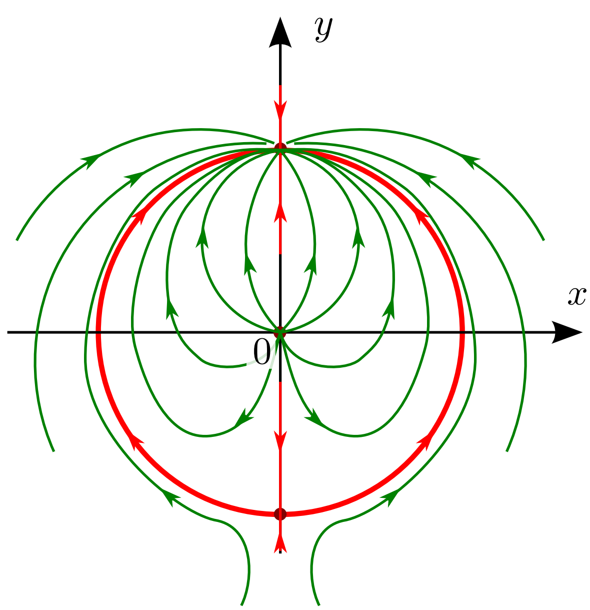

Figure 2: Phase portrait for the exercise 4.