# MAT 475: Practice midterm 2

**Exercise 1.**

The characteristic polynomial is given by: $P(\lambda) = (a-\lambda)(\lambda^2 - 2a\lambda + a^2 - bc)$. Thus the eigenvalues are given by:

$$\lambda_1 = a, \quad \lambda_{2,3} = a \pm \sqrt{bc}.$$

There are 3 real and distinct eigenvalues if $bc > 0$, a repeated eigenvalue is $bc = 0$, complex eigenvalues if $bc < 0$.

**Exercise 2.**

a) For the matrix $A$. Eigenvalues/eigenvectors:

$$\lambda_1 = 6,\ \mathbf{u}_1 = (1,2,-1),\ \lambda_{2,3} = 0, \mathbf{u}_2 = (1,0,1),\ \mathbf{u}_3 = (2,-1,0).$$

Thus, the general solution is given by:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{6t}\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3.$$

b) For the matrix $B$. Eigenvalues/eigenvectors:

$$\lambda_1 = 2,\ \mathbf{u}_1 = (1,1,1),\ \lambda_{2,3} = 1 \pm i, \mathbf{w} = (-1,-i,i).$$

Let $\mathbf{u}_2 = (-1,0,0)$ and $\mathbf{u}_3 = (0,-1,1)$. Thus, the general solution is given by:

$$\mathbf{x}(t) = c_1\mathrm{e}^{2t}\mathbf{u}_1 + \mathrm{e}^t\Big(c_2(\cos t\cdot\mathbf{u}_2 - \sin t\cdot\mathbf{v}_2) + c_3(\sin t\cdot\mathbf{u}_2 + \cos t\cdot\mathbf{v}_2)\Big).$$

c) For the matrix $C$. Eigenvalues/eigenvectors:

$$\lambda_1 = 1,\ \mathbf{u}_1 = (1,1,-1).$$

Solving $(C - 1\mathrm{Id})\mathbf{w}_1 = \mathbf{u}_1$ leads $\mathbf{w}_1 = (1,0,0)$. Then solving $(C - 1\mathrm{Id})\mathbf{w}_2 = \mathbf{w}_1$ leads $\mathbf{w}_2 = (1/2, 0, 1/2)$. The solution is given by:

$$\mathbf{x}(t) = c_1\mathrm{e}^t\mathbf{u}_1 + c_2\mathrm{e}^t(\mathbf{w}_1 + t\mathbf{u}_1) + c_3\mathrm{e}^t(\mathbf{w}_2 + t\mathbf{w}_1 + t^2/2\mathbf{u}_1)$$

**Exercise 3.**

Eigenvalues/eigenvectors:

$$\lambda_1 = 0,\ \mathbf{u}_1 = (1,1),\ \lambda_2 = 2, \mathbf{u}_2 = (1,-1).$$

Thus, let $D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, we deduce $T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. We have

$$\mathrm{e}^{tA} = T\mathrm{e}^{tD}T^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \mathrm{e}^{2t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\mathrm{e}^{2t} & 1-\mathrm{e}^{2t} \\ 1-\mathrm{e}^{2t} & 1+\mathrm{e}^{2t} \end{bmatrix}.$$

To find the solution to the non autonomous linear system, we compute:

$$\int_0^t \mathrm{e}^{-sA} f(s)\, ds \;=\; \frac{1}{2}\int_0^t \begin{pmatrix} \mathrm{e}^{-s}+\mathrm{e}^{-3s} \\ \mathrm{e}^{-s}-\mathrm{e}^{-3s} \end{pmatrix} ds = \frac{1}{2}\begin{pmatrix} \frac{1-\mathrm{e}^{-t}}{2}+\frac{1-\mathrm{e}^{-3t}}{6} \\ \frac{1-\mathrm{e}^{-t}}{2}-\frac{1-\mathrm{e}^{-3t}}{6} \end{pmatrix}.$$

Therefore the solution is given by:

$$\mathbf{x}(t) = \mathrm{e}^{tA}\mathbf{x}_0 + \frac{\mathrm{e}^{tA}}{2}\begin{pmatrix} \frac{1-\mathrm{e}^{-t}}{2}+\frac{1-\mathrm{e}^{-3t}}{6} \\ \frac{1-\mathrm{e}^{-t}}{2}-\frac{1-\mathrm{e}^{-3t}}{6} \end{pmatrix}.$$

**Exercise 4.**

The Picard iterations consist in iterating the following equation:

$$x_{n+1} = 0 + \int_0^t (-x_n(s)+s)\, ds$$

We deduce:

$$\begin{aligned} x_0(t) &= 0 \\ x_1(t) &= \int_0^t s\, ds = \frac{t^2}{2} \\ x_2(t) &= \int_0^t -s^2/2 + s\, ds = -\frac{t^3}{3!}+\frac{t^2}{2} \\ x_3(t) &= \int_0^t s^3/3! - s^2/2 + s\, ds = \frac{t^4}{4!}-\frac{t^3}{3!}+\frac{t^2}{2}. \end{aligned}$$

By induction, we can show that:

$$\begin{aligned} x_n(t) &= \frac{(-t)^n}{n!}+\cdots+\frac{t^4}{4!}-\frac{t^3}{3!}+\frac{t^2}{2} \\ &= \frac{(-t)^n}{n!}+\cdots+\frac{t^4}{4!}-\frac{t^3}{3!}+\frac{t^2}{2}-t+1 \quad +t-1. \end{aligned}$$

Therefore, when $n \to +\infty$, we have:

$$x(t) = \mathrm{e}^{-t}+t-1.$$

**Exercise 5.**

Equilibria: $x=2$, $y^2=1$. Thus, $\mathbf{x}_* = (2,1)$ and $\mathbf{x}_* = (2,-1)$.

Stability: $Df(\mathbf{x}_*) = \begin{bmatrix} 4 & 0 \\ 2x & -8y \end{bmatrix}$. We deduce that $\mathbf{x}_* = (2,1)$ is a saddle and that $\mathbf{x}_* = (2,-1)$ is a source.

Since both equilibria are hyperbolic (i.e. no zero eigenvalues), the phase portrait of the non-linear system is well approximate locally by the linearized system.

To sketch the phase portrait of the linear system, we compute the eigenvalues/eigenvectors at each equilibrium:

$$\begin{aligned} \text{at } (2,1) &: \quad \lambda_1 = 4, \mathbf{u}_1 = (1,1), \quad \lambda_2 = -8, \mathbf{u}_2 = (0,1) \\ \text{at } (2,-1) &: \quad \lambda_1 = 4, \mathbf{u}_1 = (1,-1), \quad \lambda_2 = 8, \mathbf{u}_2 = (0,1). \end{aligned}$$

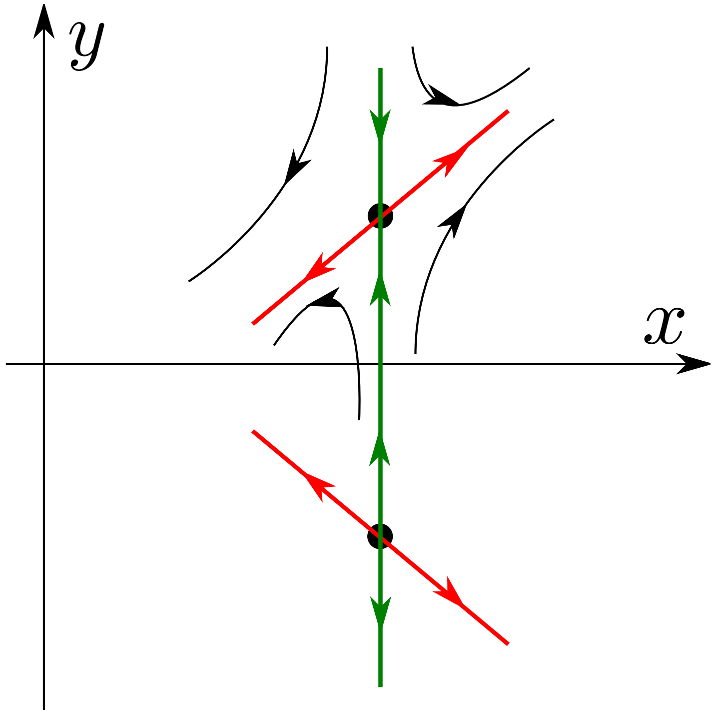


Figure 1: Phase portrait of the non-linear system near two equilibria.

We deduce the phase portrait given by figure 1.

**Exercise 6.**

Equilibria: $x = 0$, $\sin(y) = 0$. Thus, $\mathbf{x}_* = (0, n\pi)$.

Stability: $Df(\mathbf{x}_*) = \begin{bmatrix} 1 + 3\cos(3x - y) & -\cos(3x - y) \\ \mathrm{e}^x & 0 \end{bmatrix} = \begin{bmatrix} 1 + 3 \cdot (-1)^n & (-1)^{n+1} \\ 1 & 0 \end{bmatrix}$.
We deduce that if $n$ is even, then the equilibrium $\mathbf{x}_* = (0, n\pi)$ is a source, whereas the equilibrium is a saddle when $n$ is odd.

**Exercise 7.**

a) Equilibria: $x' = 0$ leads to $y = -x^2$. Thus, $y' = 0$ gives $x^2 + x + a = 0$. Therefore:

$$x_\pm = \frac{-1 \pm \sqrt{1 - 4a}}{2}.$$

Therefore, if $-\infty < a < \frac{1}{2}$, there exists two equilbria:

$$X_1 = (x_-, -(x_-)^2) \quad , \quad X_2 = (x_+, -(x_+)^2).$$

b) The linearized system is given by:

$$A := DF(X_*) = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix}.$$

At equilibrium $X_2$, we find that:

$$\det(A) = -2x - 1 = 1 - \sqrt{1 - 4a} - 1 = -\sqrt{1 - 4a} < 0,$$

thus $X_2$ is a saddle point for $a < 1/4$. At $X_1$, we find $\det(A) = \sqrt{1-4a} > 0$ for $a < 1/4$ and moreover:

$$\mathrm{Tr}(A) = 2x - 1 = -2 - \sqrt{1-4a}.$$

We deduce:

$$\mathrm{Tr}(A)^2/4 = 1 + \sqrt{1-4a} + \frac{1-4a}{4} > \sqrt{1-4a} = \det(A),$$

and we conclude that $X_1$ is a (stable) node.

c) We observe a saddle-node bifurcation at $a_* = \frac{1}{4}$.

**Exercise 8.**

Consider the dynamical system:

$$\begin{cases} x' &= x - x^3 \\ y' &= -y. \end{cases}$$

a) Equilibria: $(-1,0)$, $(0,0)$ and $(1,0)$.
We deduce:

$$\begin{aligned} DF(-1,0) &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \textit{stable node} \\ DF(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \textit{saddle} \\ DF(1,0) &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad \Rightarrow \quad \textit{stable node.} \end{aligned}$$

b) Nullclines: $x' = 0$ leads to three vertical line $x = -1$, $x = 0$ and $x = 1$. For $y' = 0$, we have the $x-$axis.

c) The phase portrait is given in figure 2.
We investigate the behavhior of solution starting at $(x_0, 1)$.

- If $x_0 < 0$, then the solution will converge to $(-1, 0)$.
- If $x_0 = 0$, then the solution will converge to $(0, 0)$.
- If $x_0 > 0$, then the solution will converge to $(1, 0)$.

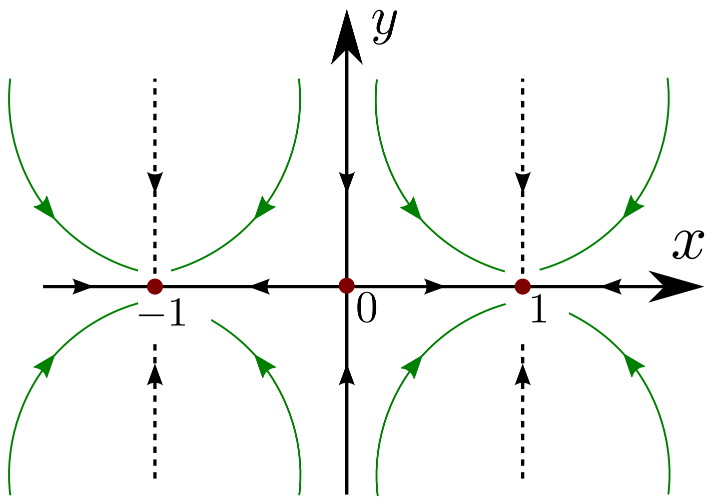


Figure 2: Phase portrait for **Ex.8**.