MAT 475: Solution homework 10 (11/27)

1 Chapter 9

Ex 8. [4pts]

a) The phase portrait is a saddle point. The dynamical system has an Hamiltonian structure with $H = xy + \frac{y^2}{2}$.

b) The dynamical system is a gradient system with $V = -xy(x + y)$. 1pt

c) It is an Hamiltonian system: $H = x^2y - xy^2$. 1pt

d) It is a gradient system: $V = -\left(\frac{x^3}{3} - x^2y + \frac{y^3}{3}\right)$. 1pt

e) It is an Hamiltonian system: $H = \sin^2 x \cos y$. To sketch the phase portrait, we draw the contour plot of $H = 0$. 1pt

2 Chapter 10

Ex 1. [3pts]

a) $r' = r(1 - r)$, $\theta' = 1$. 1pt
Denote $\mathcal{C}$ the unit circle.

- for $\mathbf{x} \neq (0, 0)$, $\omega(\mathbf{x}) = \mathcal{C}$.
- for $\mathbf{x} = (0, 0)$, $\omega(\mathbf{x}) = \{(0, 0)\}$.
- for $\mathbf{x}$ inside $\mathcal{C}$, $\alpha(\mathbf{x}) = \{(0, 0)\}$.
- for $\mathbf{x}$ on $\mathcal{C}$, $\alpha(\mathbf{x}) = \mathcal{C}$.

b) $r' = r(r - 1)(r - 2)$, $\theta' = 1$. 1pt
Denote $\mathcal{C}_1$ and $\mathcal{C}_2$ the circles of radius (resp.) 1 and 2.

- for $\mathbf{x} \neq (0, 0)$ and inside $\mathcal{C}_2$: $\omega(\mathbf{x}) = \mathcal{C}_1$.
- for $\mathbf{x}$ on $\mathcal{C}_2$, $\omega(\mathbf{x}) = \mathcal{C}_2$.
- for $\mathbf{x} = (0, 0)$, $\omega(\mathbf{x}) = \{(0, 0)\}$.

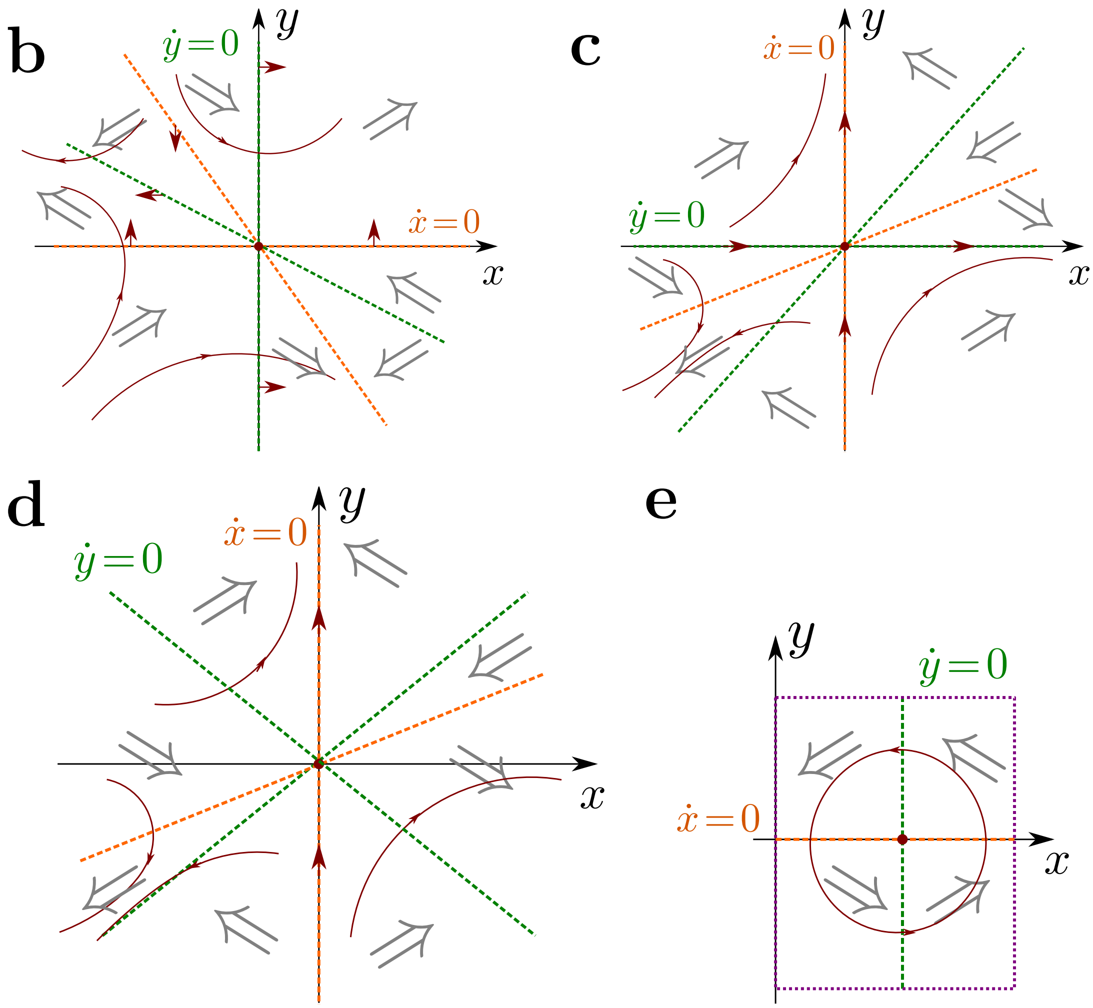

Figure 1: Phase portraits for **Ex.8**.

- for $\mathbf{x} \neq (0,0)$ outside $\mathcal{C}_1$: $\alpha(\mathbf{x}) = \mathcal{C}_2$.
- for $\mathbf{x}$ on $\mathcal{C}_1$, $\alpha(\mathbf{x}) = \mathcal{C}_1$.
- for $\mathbf{x} \neq (0,0)$ inside $\mathcal{C}_1$: $\alpha(\mathbf{x}) = \{(0,0)\}$.

c) $r' = \sin r$, $\theta' = -1$.
All the circles of radius $k\pi$ are ω-limit and α-limit.

d) $x' = \sin x \sin y$, $y' = -\cos x \cos y$. 1pt
The equilibria $(m\pi, n\pi + \pi/2)$ and $(m\pi + \pi/2, n\pi)$ are ω-limit and α-limit.

Ex 2.

We parametrize the local section by (r, z) (it is a plan). The explicit solution of $r' = r(1-r)$ is given by $r(t) = \frac{1}{(1/r_0 - 1)\mathrm{e}^{-x} + 1}$. Similarly, for z, we have: $z(t) = z_0 \mathrm{e}^{-t}$. Since $\theta' = 1$, the first crossing time for any solution and any transverse line along the unit circle is 2π. We deduce that the Poincaré map is given by:

$$P(r,z) = \left(\begin{array}{c} \frac{1}{(1/r-1)\mathrm{e}^{-2\pi}+1} \\ z\mathrm{e}^{-2\pi} \end{array} \right).$$

Therefore, the differential of P is given by:

$$DP(1,0) = \left[\begin{array}{cc} \mathrm{e}^{-2\pi} & 0 \\ 0 & \mathrm{e}^{-2\pi} \end{array} \right].$$

Thus, $|DP(1,0)| < 1$ and the periodic solution is asympotically stable.

Ex 4.

Denote $\mathcal{D}$ the square bounded by $x = 0, \pi$ and $y = 0, \pi$. Studying the nullcline along the square, we deduce that the solutions starting inside $\mathcal{D}$ cannot exit the domain.

To show that the solutions are converging toward the square, we have to use a Lyapunov function. Consider:

$$V(x,y) = \sin x \sin y.$$

We observe that V is maximal at $(\pi/2, \pi/2)$ and minimum (inside $\mathcal{D}$) at the boundary of $\mathcal{D}$. We now estimate the evolution of V along the solution:

$$\begin{aligned} \frac{d}{dt}V(x(t),y(t)) &= \cos x \sin y x' + \sin x \cos y y' \\ &= \cos x \sin y \sin x(-.1\cos x - \cos y) \; + \; \sin x \cos y \sin y(\cos x - .1\cos y) \\ &= -.1 \sin x \sin y(\cos^2 x + \cos^2 y) < 0, \end{aligned}$$

inside $\mathcal{D}$. Thus, V is decaying toward and thus the solution has to move closer to the minimum of V which is at the boundary of $\mathcal{D}$.

Ex 7. [3pts]

Let's consider S a local section. By assumption of the vector field, any solution starting on S will cross S once more at a later time (the flow keeps *rotating* the solution along A). Thus, we can consider the Poincaré map $P : S \to S$.

We parametrize the local section S inside A by the interval $[a,b]$ (see figure 2). Since the vector field is pointing inside A, we have: $P(a) > a$ and $P(b) < b$. In other words, $P([a,b]) \subset [a,b]$. Since P is continuous, this implies that P has a fixed point on $[a,b]$: there exists $c \in [a,b]$ such that $P(c) = c$. The solution starting at c is a periodic solution.

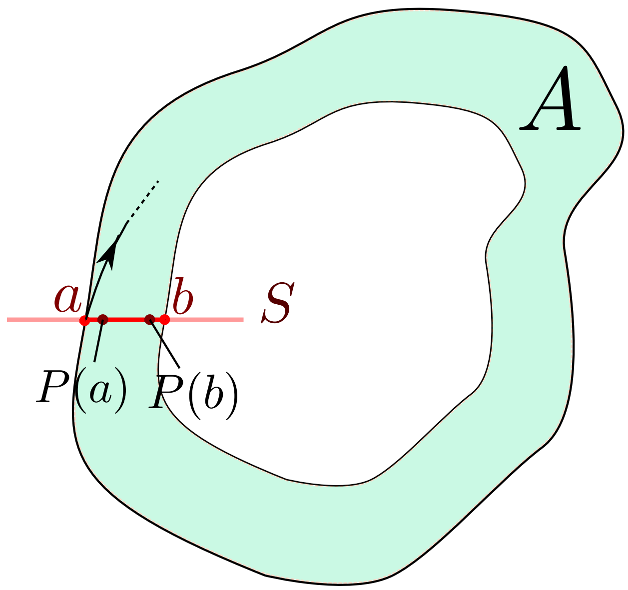

Figure 2: Illustration of the Poincaré map for **Ex 7**.