# MAT 475: Solution homework 8 (10/30)

## 1 Chapter 8

**Ex 4.**

Let assume that $f_a(x_0) = 0$ for any $a$. Thus:

$$\frac{\partial}{\partial a} f_{a_0}(x_0) = 0,\ \frac{\partial^2}{\partial a^2} f_{a_0}(x_0) = 0, \ldots, \frac{\partial^k}{\partial a^k} f_{a_0}(x_0) = 0. \qquad (1)$$

We sketch the bifurcation diagram near the pitchfork bifurcation in figure 1. Approximating the derivative in $x$ by finite difference:

$$\begin{aligned} \frac{\partial}{\partial x_0} f_{a_0}(x_0) &= \lim_{\Delta x \to 0} \frac{f_{a_0}(x_0 + \Delta x) - f_{a_0}(x_0)}{\Delta x} \\ \frac{\partial^2}{\partial x_0^2} f_{a_0}(x_0) &= \lim_{\Delta x \to 0} \frac{f_{a_0}(x_0 + \Delta x) - 2 f_{a_0}(x_0) + f_{a_0}(x_0 - \Delta x)}{\Delta x^2} \end{aligned}$$

we guess that:

$$\frac{\partial}{\partial x} f_{a_0}(x_0) = \frac{\partial^2}{\partial x^2} f_{a_0}(x_0) = 0. \qquad (2)$$

Moreover, we need to have $f_a$ is non-identically equal to zero. We also impose:

$$\frac{\partial^2}{\partial a \partial x} f_{a_0}(x_0) \neq 0, \qquad \frac{\partial^3}{\partial x^3} f_{a_0}(x_0) \neq 0. \qquad (3)$$

**Remark.** A typical example of pitchfork bifurcation is given by $f_a(x) = ax - x^3$ at $a_0 = 0$ and $x_0 = 0$. We observe that:

$$\frac{\partial}{\partial a} f_{a_0}(x_0) = \frac{\partial^2}{\partial a^2} f_{a_0}(x_0) = \frac{\partial}{\partial x_0} f_{a_0}(x_0) = \frac{\partial^2}{\partial x_0^2} f_{a_0}(x_0) = 0,$$

and $\frac{\partial^2}{\partial a \partial x} f_{a_0}(x_0) = 1 \neq 0$, $\frac{\partial^3}{\partial x^3} f_{a_0}(x_0) = -6 \neq 0$. As we will see next, this example is actually "generic" (i.e. any pitchfork bifurcation looks locally like "$c_1 ax + c_2 x^3$").

To show that the conditions (1)-(3) are sufficient, we perform a Taylor expansion of $f_a(x)$ near $a_0$ and $x_0$. Without loss of generality, let's suppose[1] $a_0 = x_0 = 0$. We obtain:

$$\begin{aligned} f_a(x) &= f_{a_0}(x_0) + \partial_a f \cdot a + \partial_x f \cdot x \\ &\quad + \frac{1}{2}\left(\partial^2_{aa} f \cdot a^2 + 2\partial^2_{ax} f \cdot ax + \partial^2_{xx} f \cdot x^2\right) + \ldots \\ &= \partial^2_{ax} f \cdot ax + \ldots \end{aligned}$$

[1] otherwise replace all the "$x$" by "$x - x_0$" and "$a$" by "$a - a_0$"

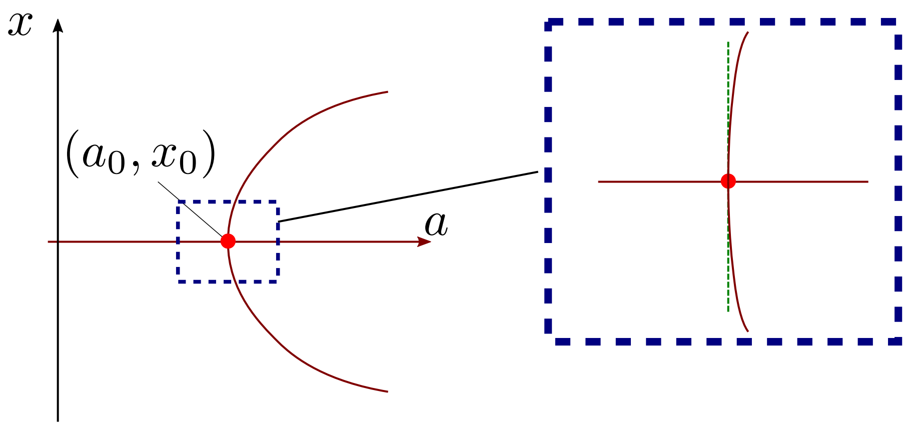


Figure 1: An example of pitchfork bifurcation. Near $(a_0, x_0)$, the zeros of $f_a(x)$ are "close" to the lines $a = a_0$ and $x = x_0$.

using the conditions (1)-(3). Therefore, near $(a_0, x_0)$ the zeros of the function $f_a(x)$ will be (approximately) along the axis $a = a_0$ and $x = x_0$. Looking at the third order terms, we obtain:

$$\begin{aligned} f_a(x) &= \partial^2_{ax} f \cdot ax + \frac{1}{3!}\left(0 + 3\partial^3_{aax} f \cdot a^2 x + 3\partial^3_{axx} f \cdot ax^2 + \partial^3_{xxx} f \cdot x^3\right) + \ldots \\ &= \partial^2_{ax} f \cdot ax + \frac{1}{6}\partial^3_{xxx} f \cdot x^3 + \mathcal{O}(|ax|) \end{aligned}$$

using condition (1). Therefore, for $a$ small enough: $f_a(x) \approx c_1 ax + c_2 x^3$ with $c_1 = \partial^2_{ax} f$ and $c_2 = \frac{1}{6}\partial^3_{xxx} f$. We deduce that $f_a(x)$ will undergo a pitchfork bifurcation at $a = 0$ and $x = 0$ if $c_1 \neq 0$ and $c_2 \neq 0$.

**Ex 8.** [3pts]

At $a = 0$, there are 5 equilibria: $(0,0)$, $(\pm 1, 0)$, $(0, \pm 1)$, and there are all unstable (see figure 2). There is no more equilibrium for $a > 0$. However, for $-1 < a < 0$, each equilibrium on the circle gives two equilibria ("saddle-node"). Indeeed, the equation $\sin^2(\theta/2) + a = 0$ has 8 solution on $[0, 2\pi)$. Therefore, we have 4 bifuractions at each equilibrium on the circle at $a = 0$.

Similarly, at $a = -1$, we observe 4 saddle-node bifurcations at the equilibria: $(\pm\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, \pm\sqrt{2})$.

**Ex 9.** [2pts]

a) We have a bifurcation at $a = 1$ and $a = -1$: there are 2 equilibria for $-1 < a < 1$ and none for $|a| > 1$.

b-d) The description is given by figure 3.

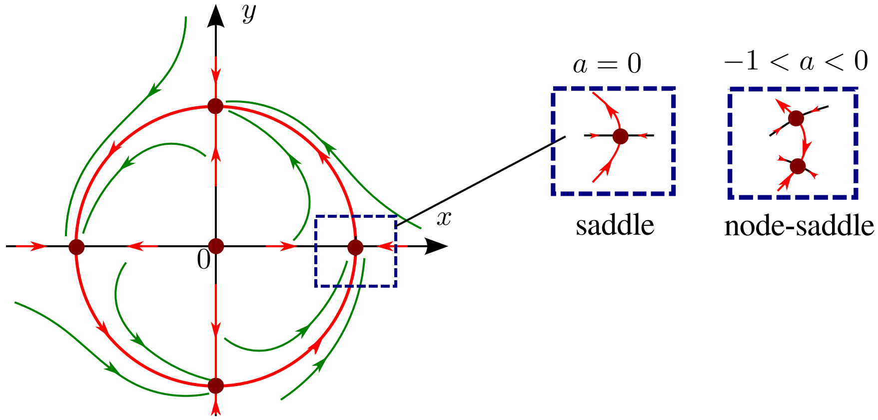


Figure 2: Phase portrait for **Ex 8** at $a = 0$ and for $-1 < a < 0$ near the equlibrium $(1, 0)$.

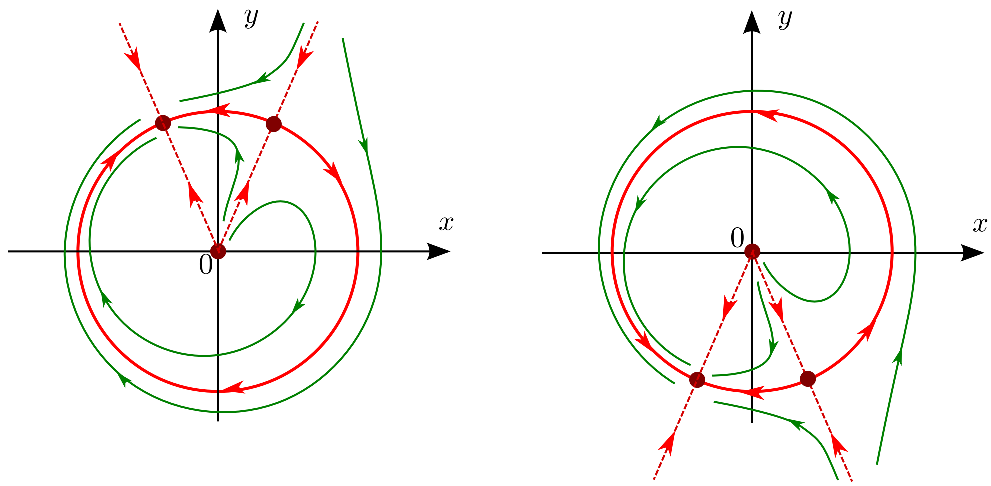


Figure 3: Phase diagram for $-1 < a < 0$ (left) and for $0 < a < 1$ (right).

## 2 Chapter 9

**Ex 1.** [5pts]

The phase portrait are given in figure 4.

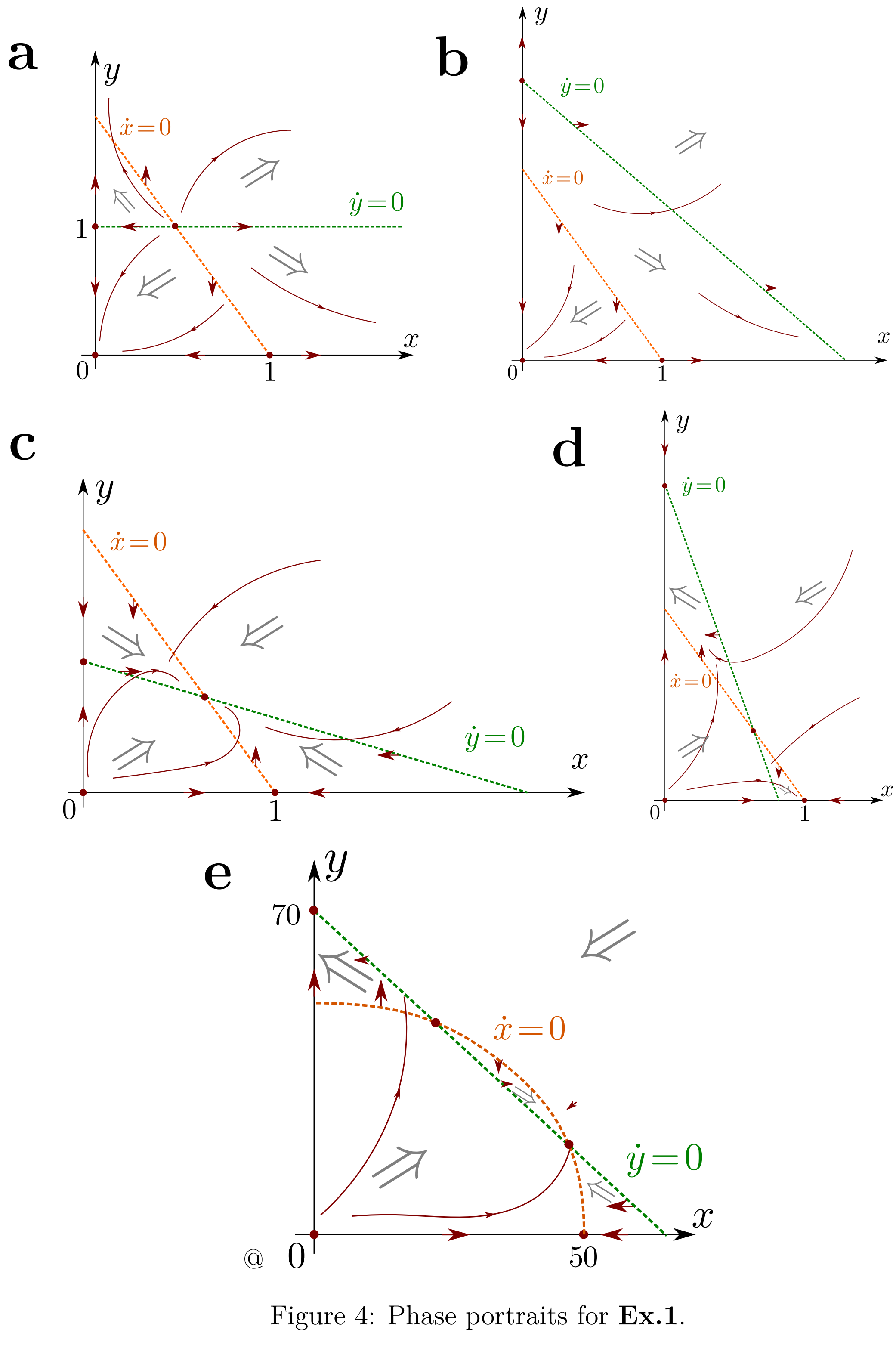


Figure 4: Phase portraits for **Ex.1**.