MAT 475: Practice final

Exercise 1.

Consider the linear system: $\mathbf{x}' = A\mathbf{x}$ with A:

$$i)\ A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad ii)\ B = \begin{bmatrix} 1 & -2 & 1 \\ 0 & -2 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad iii)\ C = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix}.$$

a) Find the general solution in each case.

b) Compute the exponential e^{tA}.

Exercise 2. For each differential equations, find the equilibria, study their stability and draw the phase line.

a) $x' = x^3 - x$.

b) $x' = 1 - |x|$.

c) $x' = \ln(1 + x^2) - 1$.

d) $x' = \sin \pi x^2$.

Exercise 3. Consider the linear system $\mathbf{x}' = A\mathbf{x}$ with the matrix $A = \begin{bmatrix} a & b \\ 1 & 1 \end{bmatrix}$.

a) Find the values of a, b for which the solutions are saddle points, nodes and spirals (indicating the stability of the origin).

b) Draw the regions in the $ab-$plane.

Exercise 4. Consider the dynamical system:

$$\begin{cases} x' &= -x - y^2 \\ y' &= -y - x^2 \end{cases}$$

a) Find the equilibria and study their stability.

b) Deduce the phase portrait near those equilibria.

c) Show that the unit ball $B = \{x^2 + y^2 \leq 1\}$ is invariant.
Hint. Use $V = x^2 + y^2$.

Exercise 5.

Consider the dynamical system:

$$\begin{cases} x' &= y \\ y' &= 2x^3. \end{cases}$$

a) Show that $H(x,y) = \frac{y^2}{2} - \frac{x^4}{2}$ is an Hamiltonian of the system.

b) Draw the contonur plot: $H(x,y) = 0$.
Deduce the stability or unstability of the origin.

Exercise 6.

Consider the dynamical system:

$$\begin{cases} x' &= 3y^2 - 3x \\ y' &= -3x^2 + 3y. \end{cases}$$

a) Find the Hamiltonian H associated with this system.

b) Deduce the stability of the equilibria.

c*) Draw the phase portrait.

Exercise 7.

Consider the dynamical system:

$$\begin{cases} x' &= 1 - x - y \\ y' &= -xy. \end{cases}$$

a) Find and study the stability of the equilibria.

b) Find and draw the nullclines.

c) Sketch the phase portrait.
What is the behavior of the solutions starting at $(0, .5)$?

Exercise 8.

Consider the dynamical system:

$$\begin{cases} x' &= 4 - x^2 - y^2 \\ y' &= 1 - xy. \end{cases}$$

a) Find and draw the nullclines.

b) Find and study the stability of the equilibria.

c) Study the behavior of the solution starting at $(1,1)$.

Exercise 9.

Consider the dynamical system:

$$\begin{cases} x' &= x - y^2 - x(x^2 + y^2) \\ y' &= y + xy - y(x^2 + y^2) \end{cases}$$

a) Study the stability at the equilibria points $(-1,0)$, $(0,0)$ and $(1,0)$.

b) Show that the dynamical system can be written in polar coordinates:

$$\begin{cases} r' &= r(1 - r^2) \\ \theta' &= r \sin\theta \end{cases}$$

c) Sketch the phase portrait.
Hint. Study the behavior of the solution starting with $r = 1$ *or* $\sin\theta = 0$.