MAT 475: Solution practice final

Exercise 1.

- For A, eigenvalues/eigenvectors:

$$\lambda_1 = -1,\ \mathbf{u}_1 = (-1,1,0),\ \lambda_2 = 1,\ \mathbf{u}_2 = (1,1,0),\ \lambda_3 = 1,\ \mathbf{u}_3 = (1,2,1).$$

  Thus, the general solution is given by:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{\lambda_1 t}\mathbf{u}_1 + c_2 \mathrm{e}^{\lambda_2 t}\mathbf{u}_2 + c_1 \mathrm{e}^{\lambda_3 t}\mathbf{u}_3.$$

  To compute the exponential, we introduce $P = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$. We deduce:

$$\mathrm{e}^{tA} = P \begin{bmatrix} \mathrm{e}^{-t} & 0 & 0 \\ 0 & \mathrm{e}^{t} & 0 \\ 0 & 0 & \mathrm{e}^{t} \end{bmatrix} P^{-1}.$$

- For B, eigenvalues/eigenvectors:

$$\lambda_{1,2} = 1 \pm i,\ \mathbf{w} = \mathbf{e}_1 \pm i\mathbf{e}_3,\ \lambda_3 = -2,\ \mathbf{u}_3 = (1,2,1).$$

  Thus, the general solution is given by:

$$\mathbf{x}(t) = \mathrm{e}^{t}\Big(c_1(\cos t\mathbf{e}_1 - \sin t\mathbf{e}_2) + c_1(\sin t\mathbf{e}_1 + \cos t\mathbf{e}_2)\Big) + c_3 \mathrm{e}^{\lambda_3 t}\mathbf{u}_3.$$

  Let $P = [\mathbf{e}_1, \mathbf{e}_3, \mathbf{u}_3]$, the exponential is given by:

$$\mathrm{e}^{tB} = P \begin{bmatrix} \mathrm{e}^{t}\cos t & -\mathrm{e}^{t}\sin t & 0 \\ \mathrm{e}^{t}\sin t & \mathrm{e}^{t}\cos t & 0 \\ 0 & 0 & \mathrm{e}^{-2t} \end{bmatrix} P^{-1}.$$

- For C, there was a typo:

$$C = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 0 \end{bmatrix}.$$

  Eigenvalues/eigenvectors:

$$\lambda_{1,2} = -1,\ \mathbf{u}_1 = (1,0,1),\ \lambda_3 = 1,\ \mathbf{u}_3 = \mathbf{e}_2.$$

  Generalized eigenvector: $\mathbf{w} = (0,1,1)$. The general solution is:

$$\mathbf{x}(t) = c_1 \mathrm{e}^{-t}\mathbf{u}_1 + c_2 \mathrm{e}^{-t}\Big(\mathbf{w} + t\mathbf{u}_1\Big) + c_1 \mathrm{e}^{\lambda_3 t}\mathbf{u}_3.$$

  Let $P = [\mathbf{u}_1, \mathbf{w}, \mathbf{e}_3]$, the exponential is given by:

$$\mathrm{e}^{tC} = P \begin{bmatrix} \mathrm{e}^{-t} & t\mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{-t} & 0 \\ 0 & 0 & \mathrm{e}^{t} \end{bmatrix} P^{-1}.$$

  c

Exercise 2.

a) Equilibria: $x = 0$ (stable), $x = \pm 1$ (unstable).

b) Equilibria: $x = -1$ (unstable), $x = 1$ (stable).

c) Equilibria: $x = -\sqrt{e - 1}$ (stable), $x = \sqrt{e - 1}$ (unstable).

d) Equilibria: $x = \sqrt{2k}$ (unstable), $x = \sqrt{2k+1}$ (stable).

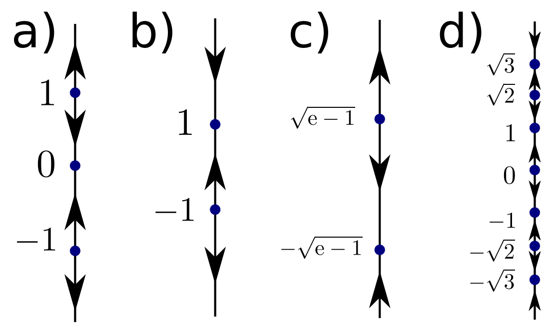

Figure 1: Phase lines for **Ex.2**.

Exercise 3.

a) Since $D = \det(A) = a - b$ and $T = \text{Trace}(A) = a + 1$, we find the regions:

- saddle: $a - b < 0$
- spiral: $a - b > \frac{1}{4}(a+1)^2 \;\Rightarrow\; b < -\frac{1}{4}(a-1)^2$.
- node: $b > -\frac{1}{4}(a-1)^2$ and $a - b > 0$
- center: $a + 1 = 0$ and $a - b > 0$.

Spiral and nodes are stable if $T < 0$ meaning $a < -1$.

b) See figure 2 for the phase diagram.

Exercise 4.

a) Equilibria: $(-1,-1)$, $(0,0)$.
Stability: $DF(x,y) = \begin{bmatrix} -1 & -2y \\ -2x & -1 \end{bmatrix}$. Thus,

$$\text{at } (-1,-1) \qquad DF(-1,-1) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \qquad \Rightarrow \textit{saddle}$$

$$\text{at } (0,0) \qquad DF(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \Rightarrow \textit{sink}.$$

b) At the equilibria $(-1,-1)$, the eigenvectors are:

$$\lambda_1 = -3,\ \mathbf{u}_1 = (1,1) \quad \lambda_2 = 1,\ \mathbf{u}_2 = (1,-1).$$

We deduce the phase portrait given in figure 3.

c) The time derivative of V along the solutions is given by:

$$\dot{V} = 2x(-x-y^2) + 2y(-y-x^2) = -2x^2(1+y) - 2y^2(1+x).$$

Thus, if $x \geq -1$ and $y \geq -1$, we have: $\dot{V} \leq 0$. In particular, if initially (x_0, y_0) is in unit ball (i.e. $V(x_0, y_0) \leq 1$), we have $\dot{V} \leq 0$ and thus $V(x(t), y(t)) \leq 1$. As a consequence, the solution is "trapped" in the region $V \leq 1$ which is precisely the unit ball.

Exercise 5.

a) Taking $H(x,y) = \frac{y^2}{2} - \frac{x^4}{2}$, we have

$$x' = \frac{\partial H}{\partial y} \quad , \quad y' = -\frac{\partial H}{\partial x}.$$

Therefore, H is an Hamiltonian for the system, the solutions are 'trapped' in the region $H(x,y) = C$ (since H is constant along solutions).

b) The contour $H(x,y) = 0$ leads

$$y^2 = x^4 \quad \Rightarrow \quad y = \pm x^2.$$

We deduce 5 solutions curves (see figure 4 in red).
We conclude that the origin is not stable as two solutions 'escape' from the origin.

Exercise 6.

Consider the dynamical system:

$$\begin{cases} x' &= 3y^2 - 3x \\ y' &= -3x^2 + 3y. \end{cases}$$

a) $H(x,y) = x^3 + y^3 - 3xy$.

b) Finding the equilibria lead to: $x = y^2$ and $y = x^2$. Thus, $y - y^4 = 0$ and therefore $y = 0$ or $y = 1$. We deduce two equilibria: $(1,1)$ and $(0,0)$.
At the origin $(0,0)$, the Jacobian is given by:

$$DF(0,0) = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus, the origin is a saddle point.
At $(1,1)$, the Jacobian is given by:

$$DF(1,1) = \begin{bmatrix} -3 & 6 \\ -6 & 3 \end{bmatrix}.$$

We cannot conclude about the stability of $(1,1)$ since we have a center for the linearized system (i.e. $(1,1)$ not hyperbolic).

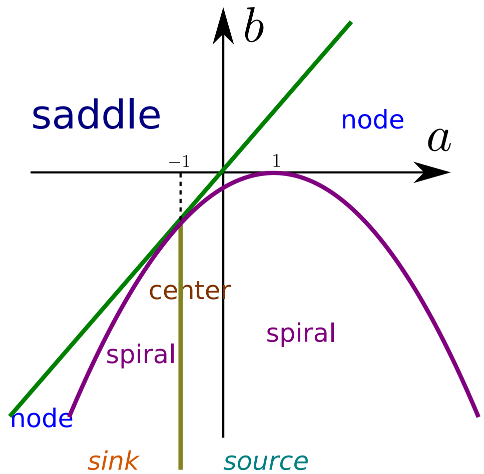

Figure 2: Phase diagram for **Ex.4**.

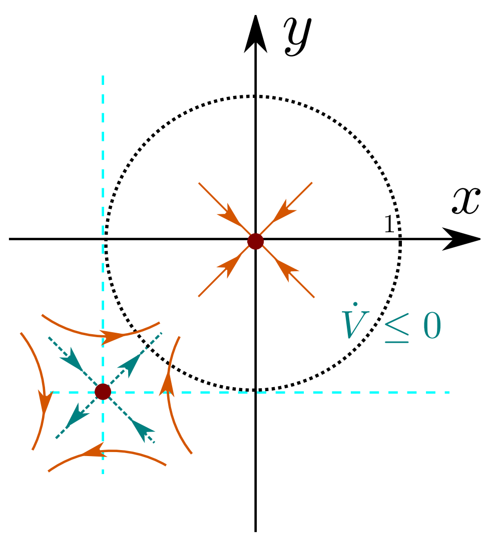

Figure 3: Phase portrait for **Ex.**5.

Thus, we look if we can use the Hamiltonian to deduce the stability. At $(1,1)$, we find $\nabla H(1,1) = 0$ and

$$\nabla^2 H(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}.$$

Since determinant and trace are positive, both eigenvalues are strictly positive. We deduce that $(1,1)$ is a strict local minimum. Therefore, H is a Lyapunov function $(1,1)$, and we conclude that $(1,1)$ is a stable equilibrium.

c*) To draw the phase portrait, we have to plot the contour plot $H(x,y) = C$. We notice that for $C = 1$, the curve is actually fhe folium of Descartes. We use this curve to deduce the behavior of the solutions nearby (see figure).

Exercise 7.

Consider the dynamical system:

$$\begin{cases} x' &= 1 - x - y \\ y' &= -xy. \end{cases}$$

a) Equilibria: $(0,1)$, $(1,0)$.
We deduce:

$$\begin{aligned} DF(0,1) &= \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} & \Rightarrow & \quad \textit{stable} \\ DF(1,0) &= \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} & \Rightarrow & \quad \textit{stablenode.} \end{aligned}$$

b-c) See figure 6.
The solution starting at $\mathbf{x}_0 = (0, .5)$ is trapped in the triangle formed by the nullclines. The solution will converges to the equilibrium $(1,0)$.

Exercise 8.

Consider the dynamical system:

$$\begin{cases} x' &= 4 - x^2 - y^2 \\ y' &= 1 - xy. \end{cases}$$

a) See figure 7.

b) Equilibria: $y = 1/x$ and $4 - x^2 - 1/x^2 = 0$ thus $x^4 - 4x^2 + 1 = 0$ which gives $x = \pm\sqrt{2 \pm \sqrt{3}}$.
Two equilibria are saddle (if $|x_*| < 1$). At $x_* = \sqrt{2+\sqrt{3}}$, the equilibrium is a stable node and at $x_* = -\sqrt{2+\sqrt{3}}$, the equilibrium is a unstable node.

c) The solution is trapped in between the two nullclines and converges toward the stable equilibrium..

Exercise 9.

a) The Jacobian is given by:

$$Df(x,y) = \begin{bmatrix} 1-(x^2+y^2)-2x^2 & -2y-2xy \\ y-2xy & 1+x-(x^2+y^2)-2y^2 \end{bmatrix}$$

Thus,

$$\begin{aligned} Df(-1,0) &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} & \quad \textit{stable node} \\ Df(0,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \quad \textit{unstable node} \\ Df(1,0) &= \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} & \quad \textit{saddle point.} \end{aligned}$$

b) In polar coordinates:

$$\begin{aligned} r' &= \frac{xx'+yy'}{r} = \frac{x^2+y^2-(x^2+y^2)(x^2+y^2)}{r} = r - r^3, \\ \theta' &= \frac{-x'y+y'x}{r^2} = \frac{-xy+y^3+xy+x^2y}{r^2} = \frac{(x^2+y^2)y}{r^2} = y = r\sin\theta. \end{aligned}$$

c) The phase portrait is given in figure 8.

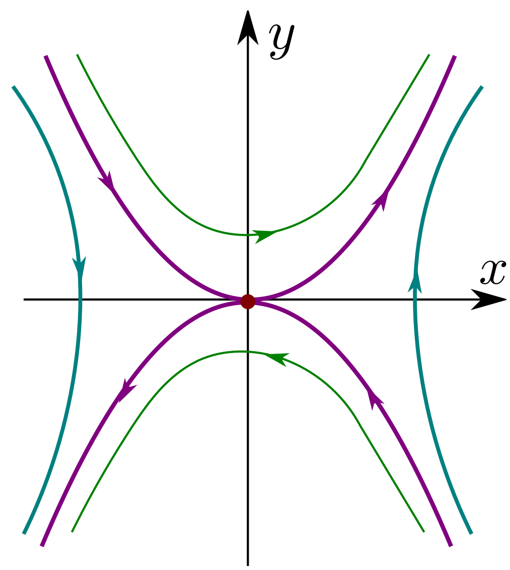

Figure 4: Phase portrait for **Ex.**5.

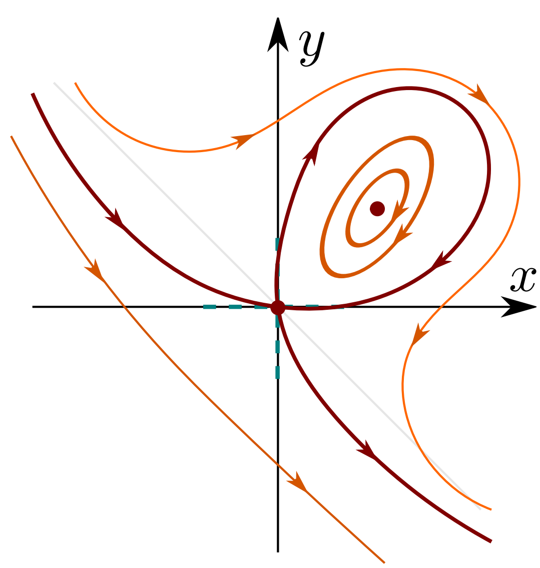

Figure 5: Phase portrait for **Ex.6**.

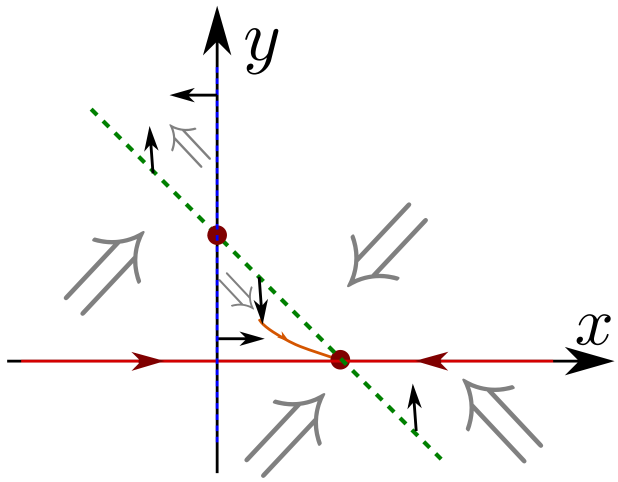

Figure 6: Phase portrait for **Ex.7**.

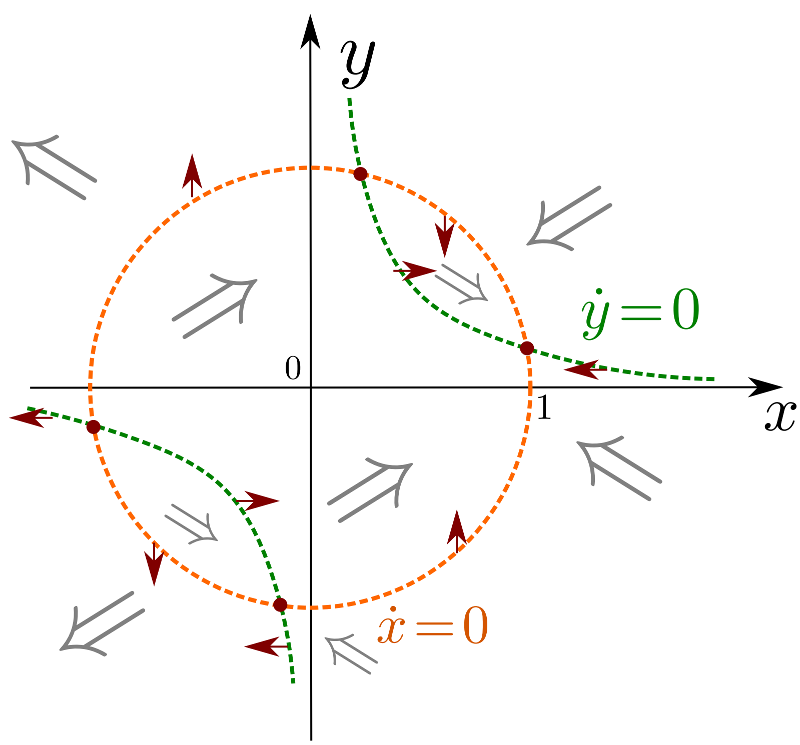

Figure 7: Phase portrait for **Ex.8**.

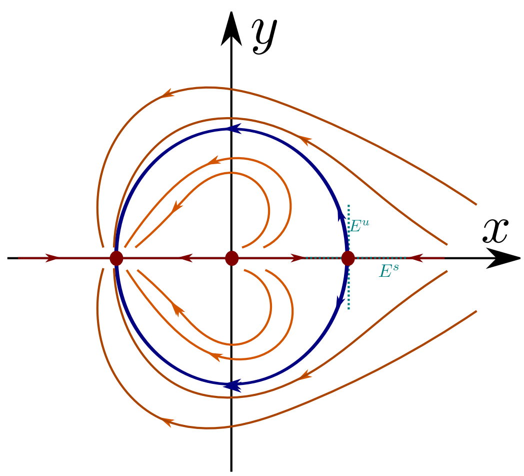

Figure 8: Phase portrait for **Ex.9**.