APM 577: Homework 1 (01/30)

1 Envelope

Ex 1. Compute the envelope of the following family of curves:

a) $u(x; a) = \frac{2}{a} - \frac{x}{a^2}$ for $x \in \mathbb{R}$ and $a \neq 0$.

b) $u(x; a) = 2a \cdot x + |a|^2$ for $a, x \in \mathbb{R}^n$.

2 Method of characteristics

Ex 2. Solve using the method of characteristics:

a) $x\partial_x u + y\partial_y u = 2u$ with $u(x, 1) = g(x)$.

b) $x\partial_x u + 2y\partial_y u + \partial_z u = 3u$ with $u(x, y, 0) = g(x, y)$.

c) $u\partial_x u + \partial_y u = 1$ with $u(x, x) = \frac{x}{2}$.

3 Hamilton-Jacobi

Ex 3. Consider the problem of minimizing the action: $\int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s))\,\mathrm{d}s$ over the function space:

$$\mathcal{A} = \{\mathbf{w} \in C^2([0, t]; \mathbb{R}^n) \text{ such that } \mathbf{w}(t) = x\}$$

without requiring $\mathbf{w}(0) = y$.

a) Show that a minimizer $\mathbf{x}(\cdot) \in \mathcal{A}$ has to solve the Euler-Lagrange equations:

$$\frac{d}{ds}\nabla_v L(\dot{\mathbf{x}}(s), \mathbf{x}(s)) = \nabla_x L(\dot{\mathbf{x}}(s), \mathbf{x}(s)) \qquad 0 \leq s \leq t.$$

b) Show moreover that $\nabla_v L(\dot{\mathbf{x}}(0), \mathbf{x}(0)) = 0$.

c) Consider now the modified action:

$$\int_0^t L(\dot{\mathbf{w}}(s), \mathbf{w}(s))\,\mathrm{d}s + g(\mathbf{w}(0)).$$

Show that a minimizer $\mathbf{x}(\cdot) \in \mathcal{A}$ satisfies the usual the Euler-Lagrange equations and determine the boundary condition at $s = 0$.

Ex 4. Let $\overline{\Omega}$ a closed subset of $\mathbb{R}^n$ and consider the initial value problem:

$$\begin{cases} u_t + |\nabla_x u|^2 = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \\ +\infty & \text{if } x \notin \overline{\Omega} \end{cases} \end{cases}$$

Show that the Hopf-Lax formula gives the solution:

$$u(x,t) = \frac{1}{4t}\Big(\text{dist}(x,\overline{\Omega})\Big)^2.$$

4 Conservation laws

Ex 5. Consider the conservation laws: $u_t + \left(\frac{u^2}{2}\right)_x = 0$. Show that:

$$u(x,t) = \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{if } 4x + t^2 > 0 \\ 0 & \text{if } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution.

Ex 6. Assume u satisfies $u(x+z) - u(x) \le Ez$ for a given E. Let $u^\varepsilon = \eta_\varepsilon * u$ where η_ε is a usual molifier. Show that:

$$u_x^\varepsilon \le E.$$

Ex 7. Compute explicitly and draw the entropy solution to:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 & \text{in } \mathbb{R} \times (0,+\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R} \end{cases}$$

with

$$u_0(x) = \begin{cases} 1 & \text{if } x \in (-\infty,-1) \\ 0 & \text{if } x \in (-1,0) \\ 2 & \text{if } x \in (0,1) \\ 0 & \text{if } x \in (1,\infty) \end{cases}$$