APM 577: Homework 2 (03/27)

1 Weak convergence

Ex 1. Prove the weak convergence of the sequences in $L^2(0,1)$

a) Let $u_n(x) = \sin(n \cdot x)$. Show $u_n \rightharpoonup 0$.

b) Fix $a, b \in \mathbb{R}$ and

$$u_n(x) = \begin{cases} a & j/n \leq x < (j+1/2)/n \\ b & (j+1/2)/n \leq x < (j+1)/n \end{cases} \quad \text{for } 0 \leq j \leq n-1$$

Show $u_n \rightharpoonup \frac{a+b}{2}$.

2 Euler-Lagrange

Ex 2. Consider the PDE:

$$-\Delta u + \nabla\phi \cdot \nabla u = f \qquad \text{in } \Omega,$$

with ϕ smooth function. Find L such that the equation above corresponds to the Euler-Lagrange eq. of the functional $I[\omega] = \int_\Omega L(\nabla\omega, \omega, x)\,\mathrm{d}x$.

3 Minimizer

Ex 3. Consider the functional:

$$I[w] = \int_\Omega (1 + |\nabla w|^2)^{1/2}\,\mathrm{d}x$$

with the constraints $\mathcal{A} = \{w \in W^{1,1}(\Omega),\ w = g \text{ on } \partial\Omega\}$. Show that the usual method to prove existence of a minimizer does not work.

Ex 4. Consider the elliptic PDE:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} + \beta(u) = 0 & \text{on } \partial\Omega \end{cases}$$

with $\beta : \mathbb{R} \to \mathbb{R}$ smooth and satisfying $0 < a \leq \beta'(z) \leq b$ for any z.

a) Write down the weak formulation of this nonlinear boundary-value problem

b) Prove that there exists a unique solution.

Ex 5. Assume $f \in L^2(\Omega)$. Prove the dual variational principle:

$$\min_{w \in H_0^1(\Omega)} \int_\Omega \frac{1}{2}|\nabla w|^2 - fw \,\mathrm{d}x = \max_{\vec{\xi} \in L^2(\Omega,\mathbb{R}^n),\, \nabla\cdot\vec{\xi}=f} -\frac{1}{2}\int_\Omega |\vec{\xi}|^2 \,\mathrm{d}x.$$

Ex 6. Consider the usual functional:

$$I[w] = \int_\Omega \frac{1}{2}|\nabla w|^2 - fw \,\mathrm{d}x$$

with $f \in L^2(\Omega)$ and the constraints:

$$\mathcal{A} = \{w \in H_0^1(\Omega),\ |\nabla w| \le 1 \text{ a.e.}\}.$$

a) Show that there exists a unique minimizer $u \in \mathcal{A}$.

b) Prove that for any $v \in \mathcal{A}$:

$$\int_\Omega \nabla u \cdot \nabla(v-u) \,\mathrm{d}x \ge \int_\Omega f(v-u) \,\mathrm{d}x.$$

Hints

1) a) use Parseval's identity.

2) $\Delta u - \nabla\phi \cdot \nabla u = e^{\phi}\nabla(e^{-\phi}\nabla u)$

3) L^1 (or $W^{1,1}$) is *different* than L^p (or $W^{1,p}$) with $p > 1$.

4) the function $B(z) = \int_0^z \beta(s)\,\mathrm{d}s$ is uniformly elliptic and satisfies $B(z) \geq az^2 - cz$ for some constant c. Deduce that the functional:

$$I[w] = \int_\Omega \frac{1}{2}|\nabla w|^2 - fw\,\mathrm{d}x + \int_{\partial\Omega} B(w)\,\mathrm{d}S$$

is coercive on $H^1(\Omega)$.

5) consider $\vec{\xi} = -\nabla u$ with u minimizer of I.