MAT 423: Homework 1 (08/27)

Ex 1.

Consider the function $f(x) = x\ln(x+2)$ near $x = 0$.

a) **Point-wise estimation**.
∘ Find the Taylor approximation $\mathbf{P_2(x)}$ at $x = 0$.
∘ Deduce an approximation of $\mathbf{f(.5)}$ and find an upper bound for the error.

b) **Integral estimation**.
∘ Find $\mathbf{c} > 0$ such that:

$$|f(x) - P_2(x)| \leq \mathbf{c}|x^3| \qquad \text{for any } x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

∘ Deduce an approximation of $\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathbf{f(x)}\,\mathrm{d}\mathbf{x}$ using P_2 and find an upper-bound for the error.

Ex 2.

Use a Taylor expansion to find an approximation of $\exp(-.1)$ with accuracy 10^{-6}.

Ex 3*. [Lipschitz condition]

A function f is *Lipschitz* on an interval $[a, b]$ if there exists a constant $\mathbf{L}$ (called Lipschitz constant) such that:

$$|f(y) - f(x)| \leq \mathbf{L}|y - x| \qquad \text{for any } x, y \in [a, b].$$

a) **Lipschitz and continuity.**
∘ Show that if f is Lipschitz on $[a, b]$ then f is continuous on $[a, b]$.
∘ Find an example of a continuous function on $[0, 1]$ that is not Lipschitz.

b) **Lipschitz and derivative.** Suppose that $f \in C^1([a, b])$.
∘ Show that f is Lipschitz on $[a, b]$.
Hint. Use f' and the mean-value-theorem to find $\mathbf{L}$.
∘ Conversely, show that if f is Lipschitz on $[a, b]$ with Lipschitz constant $\mathbf{L}$, then $|f'(x)| \leq \mathbf{L}$ on $[a, b]$.

Remark. We deduce from **a)** that Lipschitz is a "stronger" assumption than continuous, and from **b)** that the Lipschitz constant $\mathbf{L}$ is the maximum of f'.

Ex 4.

Use the bisection method to find an approximation of $\sqrt{3}$ with accuracy 10^{-4}.
Hint: consider the function $f(x) = x^2 - 3$.

Ex 5.

Implement the bisection method with the numerical software of your choice.