MAT 423: Homework 3 (09/12)

Ex 1.

Consider the linear system with two parameters α and β:

$$\left\{\begin{array}{rcrcrcl} x_1 & + & 2\,x_2 & - & x_3 & = & 0 \\ & - & x_2 & + & x_3 & = & 0 \\ 3\,x_1 & + & 3\,x_2 & + & \alpha\,x_3 & = & \beta \end{array}\right.$$

For which values of α and β does the system have:

a) a unique solution, b) no solution, c) multiple solutions ?

Extra: give a geometric interpretation of the results.

Ex 2.

a) Implement the Gauss elimination method to solve:

$$A\mathbf{x} = \mathbf{b}$$

where A is a square $n \times n$ matrix and $\mathbf{b}$ a vector in $\mathbb{R}^n$. In other words, with the numerical software of your choice (e.g. Python, Julia, Matlab/Octave...), write a function taking A and b as input and return as output the solution x using Gauss elimination.

b) Estimate (numerically) the number of operations and the computational time the algorithm used for various size n.

Ex 3.

We are interested in the following boundary value problem, the goal is to find a function $u(x)$ satisfying

$$\left\{\begin{array}{ll} \frac{d^2}{dx}u(x) = 1, & 0 < x < 1 \\ u(0) = u(1) = 0. \end{array}\right. \tag{1}$$

a) Find the explicit solution $u(x)$.

b) In general, there is no explicit solution to boundary value problem. To solve numerically equation (1), fix $N > 0$ and $\Delta x = \frac{1}{N}$ and denote (see figure 1):

$$u_i = u(x_i) \quad \text{with} \quad x_i = i \cdot \Delta x \quad \text{for } i = 0 \ldots N. \tag{2}$$

Using the finite difference approximation:

$$\frac{d^2}{dx}u(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}, \tag{3}$$

we deduce the linear system:

$$\begin{cases} u_{i+1} - 2u_i + u_{i-1} = \Delta x^2 & \text{for } i = 1\dots(N-1) \\ u_0 = u_N = 0. \end{cases} \tag{4}$$

- Write down the system (4) in the form $A\mathbf{u} = \mathbf{b}$ with $\mathbf{u} = (u_1, \dots, u_{N-1})$.
- Find the solution for $N = 10$ using **Ex 2**.

c*) Estimate the error of the method: $|u_i - u(x_i)| = \mathcal{O}(\Delta x^d)$ where u_i is the numerical solution obtained from b) and $u(x_i)$ the exact solution from a).

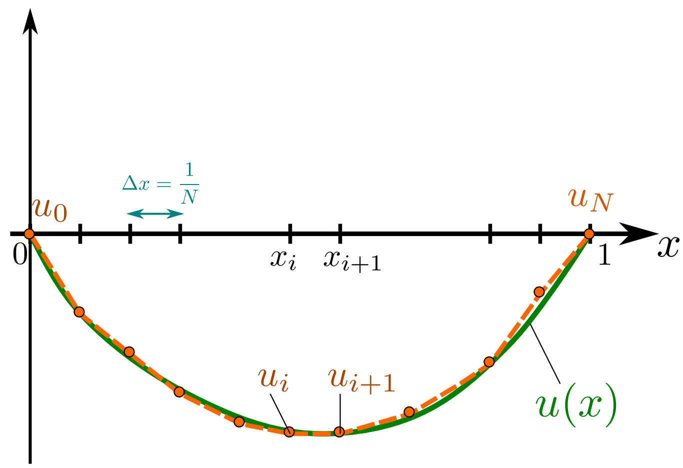

Figure 1: Discretization of the function $u(x)$ by a discrete set of points $\{u_i\}_{0\leq i\leq N}$ on the interval $x \in [0, 1]$.

Ex 4.

a) Implement the Gauss elimination with partial pivoting, i.e. at the j^{th} step in the Gauss elimination use as pivot $a_{i_0,j}$ satisfying

$$a_{i_0,j} = \max_{j\leq i\leq N} |a_{ij}|.$$

b) Use both Gauss elimination and Gauss elimination with partial pivoting to solve:

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 3x_1 + 6x_2 + 2x_3 = 1 \\ x_1 - 2x_2 + 3x_3 = 0 \end{cases}$$

What do you observe?