MAT 423: Practice midterm exam

Exercise 1. The goal of this exercise is to estimate numerically $\sqrt{2}$.

a) Use bisection method to estimate $\sqrt{2}$ and give the number of iterations required to have an accuracy of 10^{-4}.
Hint: consider $f(x) = x^2 - 2$.

b) Consider $g(x) = 1 + x - \frac{x^2}{2}$. Show that for any $x_0 \in [1, \frac{3}{2}]$, the sequence $x_{n+1} = g(x_n)$ converges to $\sqrt{2}$.

c) Apply the Newton method to estimate $\sqrt{2}$. Give a starting point x_0 and estimate the first iteration x_1.

Exercise 2.
Solve the following linear systems:

$$i) \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 3x_2 + 2x_3 = 1 \\ 3x_1 + 2x_2 + x_3 = 0 \end{cases} \qquad ii) \begin{cases} x_1 + 2x_2 = 2 \\ 2x_2 + 3x_3 = 0 \\ x_1 + 2x_2 + x_3 = 0 \end{cases}$$

Exercise 3.
Find the LU decomposition of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

Does there exist a Cholesky decomposition for A?

Exercise 4. Consider a square matrix A with its LU decomposition, i.e. $A = LU$ with L lower triangular with 1 on its diagonal and U upper-triangular.

a) Show that $\det(U) = \det(A)$.

b) Do A and U have the same eigenvalues?

Exercise 5.

a) Suppose A symmetric matrix. Show that an eigenvalue λ of A is always real.

b) Assume now that A is also definite positive. Show that the eigenvalue of A are strictly positive.

Exercise 6.
A square matrix N is called nilpotent if there exists an integer k such that: $N^k = 0$.

a) Show that the matrix $N = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$ is nilpotent.

b) Show that the eigenvalues λ of **any** nilpotent matrix N are always equal to zero.

Exercise 7.

a) Consider the matrix $A = \begin{bmatrix} -2 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$. Compute $|||A|||_1$ and $|||A|||_\infty$.

b) Compute A^{-1} and estimate $|||A^{-1}|||_1$ and $|||A^{-1}|||_\infty$.

c) Show that for **any** matrix norm $|||.|||$ and **any** non-singular matrix A:

$$|||A^{-1}||| \geq |||A|||^{-1}.$$

Exercise 8.

Consider the matrix: $A = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$. Discuss whether the Jacobi and Gauss-Seidel methods could be applied to A.

Extra. Compute the eigenvalues of T_ω the matrix associated with the SOR method.

Exercise 9.

Same as **Ex. 8** with: $B = \begin{bmatrix} 3 & -1 & -1 \\ 0 & 1 & 0 \\ -3 & 0 & 3 \end{bmatrix}$.