# MAT 423: Homework 7 (10/29)

**Ex 1.**

Consider the function: $G(x,y) = \begin{pmatrix} \frac{y}{3}\cos x \\ \frac{1}{4}\mathrm{e}^{-x^2-y^2} \end{pmatrix}$.

Show that $G$ has a fixed point.

*Hint. Show $G$ contraction on $\Omega = [-1,1]\times[-1,1]$.*

**Ex 2.**

Consider the sequence defined iteratively by $(x_0, y_0) = (1,1)$ and:

$$\begin{aligned} x_{k+1} &= \frac{1}{2}\Big(\ln(1+x_k^2) + y_k\Big) \\ y_{k+1} &= \frac{1}{10}y_k(x_k^2+y_k^2) \end{aligned}$$

Show that the sequence convergences to zero, i.e. $(x_k, y_k) \overset{k\to+\infty}{\longrightarrow} (0,0)$.

*Hint. Show and use that $G(x,y) = \Big(\frac{1}{2}(\ln(1+x^2)+y),\ \frac{1}{10}y(x^2+y^2)\Big)$ contraction on $\Omega = [0,1]\times[0,1]$.*

**Ex 3.**

Consider the function:

$$F(x,y) = \begin{pmatrix} x^3 - 3xy^2 + 1 \\ 3x^2y - y^3 \end{pmatrix}$$

a) Find a point $(x_*, y_*)$ such that $F(x_*, y_*) = (0,0)$.

b) Pick $(x_0, y_0) = (-0.8,\ 0.2)$. Apply the Newton method to estimate $(x_1, y_1)$:

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - [DF(\mathbf{x}^{(0)})]^{-1}F(\mathbf{x}^{(0)}).$$

*Extra)* Estimate numerically the convergence rate of the Newton method, i.e. find $\alpha$, $\beta$ such that:

$$\frac{\|F(\mathbf{x}^{(k+1)})\|}{\|F(\mathbf{x}^{(k)})\|^{\alpha}} \approx \beta.$$

**Ex 4.**

Consider the function $J(x,y) = 2x^2 + y^2 + (x^2+1)\sin 4y$.

Compute the first iteration of the Newton's method applied to $\nabla J$ starting at $(x_0, y_0) = (0,0)$.

**Ex 5.**

Implement numerically the quasi-Newton method to solve Ex 3.

*Extra. Estimate the order of convergence.*

## Quasi-Newton method (Broyden)

Consider $F : \mathbb{R}^n \overset{C^1}{\to} \mathbb{R}^n$.

a) Initialization:

- Pick $\mathbf{x}^{(0)}$ and define $A^{(0)} = DF(\mathbf{x}^{(0)})$.
- Define the first term:

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - [DF(\mathbf{x}^{(0)})]^{-1} F(\mathbf{x}^{(0)}).$$

b) Denote $\Delta\mathbf{x} = \mathbf{x}^{(1)} - \mathbf{x}^{(0)} \quad , \quad \Delta F = F(\mathbf{x}^{(1)}) - F(\mathbf{x}^{(0)})$.

- $\mathbf{u}^{(1)} = \frac{\Delta F - A^{(0)}\Delta\mathbf{x}}{\|\Delta\mathbf{x}\|^2}$
- $A^{(1)} = A^{(0)} + \mathbf{u}^{(1)} \otimes \Delta\mathbf{x}$
- $\mathbf{x}^{(2)} = \mathbf{x}^{(1)} - [A^{(1)}]^{-1} F(\mathbf{x}^{(1)})$.

c) Iterate: go back to b) with $\mathbf{x}_1$, $\mathbf{x}_2$ and $A^{(1)}$.

The inverse of $A^{(1)}$ should be computed using the formula:

$$(A + \mathbf{u} \otimes \mathbf{v})^{-1} = A^{-1} - \frac{A^{-1}\,\mathbf{u} \otimes \mathbf{v}\,A^{-1}}{1 + \langle A^{-1}\mathbf{u}, \mathbf{v}\rangle}$$

where $\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^T$.