MAT 423: Midterm 1 Oct. 3, 2018 Duration: 1:15

Exercise 1. 30pts + 5pts

a) Let $f(x) = x^2 - x - 1$ continuous function. Take $a = 1$ and $b = 2$. We have $f(a)f(b) = -1 < 0$. Thus, we can apply the bisection method to find a zero x_* of f on $[a, b]$. In order to have an accuracy 10^{-5}, a sufficient condition is to have the number of iteration n satisfying: 2pts 3pts

$$\frac{1}{2^{n+1}}|b-a| \leq 10^{-5} \quad \Rightarrow \quad -(n+1)\cdot\log 2 \leq -5\cdot\log 10 \quad \Rightarrow \quad n \geq 5\frac{\log 10}{\log 2} \approx 16.6.$$ 4+2pts

Thus, $n = 17$ iterations are sufficient.

b) We use the Newton method starting at $x_0 = 1$. The first iteration gives:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{1} = 2.$$ 5+3pts

c) We show that $g(x) = \sqrt{1+x}$ is a contraction on $I = [0, \infty)$.

- "$g(I) \subset I$".
  For $x \geq 0$, $g(x) = \sqrt{1+x} \geq 0$. Thus $g(x) \in I$. 4pts
- "$|g(x) - g(y)| \leq k|x - y|$".
  We estimate the maximum slope of g:

$$k = \max_{x\in I}|g'(x)| = \max_{x\geq 0}\frac{1}{2\sqrt{1+x}} = \frac{1}{2} < 1.$$ 4pts

We deduce by the mean-value theorem that g is a contraction on I. Thus, by the contracting mapping theorem, the sequence $x_{n+1} = g(x_n)$ with $x_0 \in I$ converges to the unique fixed point x_* of g on I:

$$g(x_*) = x_* \quad \Rightarrow \quad \sqrt{1+x_*} = x_* \quad \Rightarrow \quad 1 + x_* = x_*^2.$$ 3pts

Therefore $f(x_*) = 0$. Since $x_* \geq 1$, $x_* = \varphi$.

Extra: Assume g contraction on I. First, $g(g(I)) \subset g(I) \subset I$. Moreover, for any x, y in I: 2pts

$$|g(g(x)) - g(g(y))| \leq k|g(x) - g(y)| \leq k^2|x - y|$$ 3pts

with $k^2 < 1$ (since $0 \leq k < 1$). Therefore $x \mapsto g(g(x))$ is also a contraction on I.

Remark. Notice that the reverse is false: $g(x) = \mathrm{e}^{-x}$ is a not a contraction on $I = \mathbb{R}$, whereas $g(g(x)) = \mathrm{e}^{-\mathrm{e}^{-x}}$ is.

Exercise 2. 25pts + 5pts

First, we perform Gauss elimination on A:

$$\left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ 2 & 4 & 4 & 0 & 1 & 0 \\ 2 & 4 & 6 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 2 & 2 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right]$$

10pts

Thus, $MA = U$ with:

$$U = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

We then compute the inverse of M:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

8pts

Thus, $A = LU$ with $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

To solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (1, 0, 0)$, we first solve: $L\mathbf{y} = \mathbf{b}$ which gives:

$$y_1 = 1,\ y_2 = -1,\ y_3 = 0.$$

Then, $U\mathbf{x} = \mathbf{y}$ which gives:

$$x_3 = 0,\ x_2 = -\frac{1}{2}\ , x_3 = 1.$$

Thus, $\mathbf{x} = (1,\ -3\frac{1}{2}, 0)^T$.

7pts

Extra: to find the Choleski decomposition of A, we only need to change L and U by the factor $\sqrt{2}$ since $U = 2 \cdot L^T$:

$$A = LU = L \cdot 2 \cdot L^T = (\sqrt{2}L) \cdot (\sqrt{2}L)^T.$$

Thus, $A = CC^T$ with:

$$C = \sqrt{2}L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix}.$$

5pts

Exercise 3. 20pts

- Taking a vector $\mathbf{x}$ of unit length (i.e. $\|\mathbf{x}\| = 1$), we have:

$$\|AB\mathbf{x}\| = \|A(B\mathbf{x})\| \leq |||A||| \cdot \|B\mathbf{x}\|$$ 3pts

using the inequality $\|A\mathbf{z}\| \leq |||A||| \cdot \|\mathbf{z}\|$ for any vector z (notice that $B\mathbf{x}$ is not necessarily of unit length). Applying this inequality once more leads to:

$$\|AB\mathbf{x}\| \leq |||A||| \cdot \|B\mathbf{x}\| \leq |||A||| \cdot |||B||| \cdot \|\mathbf{x}\| = |||A||| \cdot |||B|||,$$ 3pts

since $\|\mathbf{x}\| = 1$. We conclude:

$$|||AB||| = \max_{\mathbf{x}=1} \|AB\mathbf{x}\| \;\leq\; \max_{\mathbf{x}=1} |||A||| \cdot |||B||| \;=\; |||A||| \cdot |||B|||.$$

- Taking $A = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$, we find $\rho(A) = \rho(B) = 1$. However, we also have: 6pts

$$AB = \begin{bmatrix} 1 & b \\ a & ab+1 \end{bmatrix}.$$

with the characteristic polynomial: $P(\lambda) = \lambda^2 - (ab+2)\lambda + 1$. Taking for example $ab = 1$, we obtain the eigenvalues: $\lambda_1 = \frac{3-\sqrt{3^2-4}}{2}$, $\lambda_2 = \frac{3+\sqrt{3^2-4}}{2} > 1$. Thus, here, $\rho(AB) > \rho(A) \cdot \rho(B)$. 3pts 5pts

Exercise 4. 25pts

- The iteration matrix for the Jacobi method is given by:

$$J_j = D^{-1}(L+U) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2/5 & 0 \end{bmatrix}.$$ 3+3pts

The characteristic polynomial of J_j is given by: $P_{J_j}(\lambda) = \lambda^2 - 4/5$. Thus, the eigenvalues are $\lambda = \pm\sqrt{\frac{4}{5}}$. Since $\rho(J_j) = \sqrt{\frac{4}{5}} < 1$, the Jacobi method converges. 4pts

- The iteration matrix for the Gauss-Seidel method is given by:

$$J_g = (D-L)^{-1}U = \begin{bmatrix} 1 & 0 \\ -2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4/5 \end{bmatrix}.$$ 3+4pts

The eigenvalues are $\lambda = 0, \frac{4}{5}$, thus $\rho(J_g) = \frac{4}{5} < 1$, the Gauss-Seidel method converges. 4pts

Moreover, since $\rho(J_g) < \rho(J_j)$, we expect the Gauss-Seidel method to converge faster than the Jacobi method. 4pts