MAT 423: Solution practice midterm

Exercise 1.

a) We consider $f(x) = x^2 - 2$ and try to find a zero of f. Consider $a = 1$ and $b = 2$, we have $f(a) \cdot f(b) < 0$. Thus we can apply the bisection method. To have an accuracy of 10^{-4}, the number of iterations n has to satisfy:

$$\frac{1}{2^{n+1}}|b-a| < 10^{-4} \quad \Rightarrow \quad n+1 > \frac{4\ln 10}{\ln 2} \quad \Rightarrow \quad n \geq 13.$$

b) Consider $g(x) = 1 + x - \frac{x^2}{2}$ and $I = [1, \frac{3}{2}]$. We want to show that g is a contraction on I.

- "$g(I) \subset I$".
  We notice that $g'(x) = 1 - x \leq 0$ on the interval I and thus g is a decreasing function on I. We deduce that for any point x on the interval I:

$$g\left(\frac{3}{2}\right) \leq g(x) \leq g(1) \quad \Rightarrow \quad 1 + \frac{3}{2} - \frac{9}{8} \leq g(x) \leq 2 - \frac{1}{2} \quad \Rightarrow \quad 1 \leq g(x) \leq \frac{3}{2}.$$

  Therefore $g(x) \in I$.
- "$|g(x) - g(y)| \leq k|x - y|$".
  We estimate the maximum slope of g:

$$k = \max_{x \in I} |g'(x)| = \max_{1 \leq x \leq \frac{3}{2}} |1 - x| = \frac{1}{2} < 1.$$

  We deduce by the mean-value theorem that g is a contraction on I. Thus, the sequence $x_{n+1} = g(x_n)$ with $x_0 \in I$ converges to the unique fixed point x_* of g (and this fixed point is $x_* = \sqrt{2}$).

c) We use once again $f(x) = x^2 - 2$ and start at $x_0 = 1$. The first iteration of the Newton method gives:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{2} = \frac{3}{2}.$$

Exercise 2.

i) Using the tableau method, we obtain:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 1 \\ 0 & -4 & -8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 1 \\ 0 & 0 & 8 & -4 \end{array}\right]$$

Thus, $x_3 = \frac{-4}{8} = -\frac{1}{2}$, $x_2 = -1 - 4x_3 = 1$, $x_1 = -2x_2 - 3x_3 = -\frac{1}{2}$.

ii) Similarly

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 2 & 3 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Thus, $x_3 = -2$, $x_2 = 3$, $x_1 = -4$.

Exercise 3.

We first perform the Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -4 & -2 & 1 & 0 \\ 0 & -4 & -8 & -3 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 8 & 5 & -4 & 1 \end{array}\right]$$

We deduce that $MA = U$ with:

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 8 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix}$$

We now need to compute the inverse of M:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 5 & -4 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -4 & 1 & -5 & 0 & 1 \end{array}\right]$$
$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 4 & 1 \end{array}\right].$$

Therefore, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$.

The matrix A is symmetric but not definite positive, thus A does not have a Choleski decomposition.

Exercise 4.

a) Assuming $A = LU$. Then $\det(A) = \det(L) \cdot \det(U) = \det(U)$ since $\det(L) = 1$ (L is lower triangular with only 1 on its diagonal).

b) A and U do not have necessarily the same eigenvalues. Take for instance:

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad , \quad U = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad A = LU = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}.$$

The eigenvalues of U are $\lambda = 2$. However, the characteristic polynomial of A is $P_A(\lambda) = \lambda^2 - 5\lambda + 4$ and $P_A(2) \neq 0$. Thus $\lambda = 2$ is not an eigenvalue of A.

Exercise 5.

a) Assume A symmetric matrix and let $(\lambda, \mathbf{u})$ eigenvalue/eigenvector of A, i.e. $A\mathbf{u} = \lambda\mathbf{u}$. We have:

$$\begin{aligned} \langle A\mathbf{u}, \mathbf{u}\rangle &= \langle \lambda\mathbf{u}, \mathbf{u}\rangle = \lambda\|\mathbf{u}\|_2^2 \\ \langle A\mathbf{u}, \mathbf{u}\rangle &= \langle \mathbf{u}, A\mathbf{u}\rangle = \langle \mathbf{u}, \lambda\mathbf{u}\rangle = \bar{\lambda}\|\mathbf{u}\|_2^2, \end{aligned}$$

where $\bar{\lambda}$ is the complex conjugate of λ. Since $\lambda = \bar{\lambda}$, λ is necessary real.

b) If moreover A is also definite positive, we obtain:

$$\langle A\mathbf{u}, \mathbf{u}\rangle > 0 \quad \Rightarrow \quad \lambda\|\mathbf{u}\|_2^2 > 0,$$

therefore $\lambda > 0$ since $\|\mathbf{u}\|_2 > 0$ (an eigenvector cannot be a zero vector).

Exercise 6.

a) $N^2 = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus N is a nilpotent matrix with $k = 2$.

b) Consider $(\lambda, \mathbf{u})$ eigenvalue/eigenvector of N. Then:

$$N^2\mathbf{u} = N(N\mathbf{u}) = N(\lambda\mathbf{u}) = \lambda N\mathbf{u} = \lambda^2\mathbf{u}.$$

More generally, $N^k\mathbf{u} = \lambda^k\mathbf{u}$. Using that N is a nilpotent matrix, we find:

$$N^k\mathbf{u} = \lambda^k\mathbf{u} \quad \Rightarrow \quad \mathbf{0} = \lambda^k\mathbf{u}.$$

Since $\mathbf{u}$ is a non-zero vector, we have $\lambda^k = 0$ and therefore $\lambda = 0$.

Exercise 7.

a) $|||A|||_1 = \max(3,3,3) = 3$, $|||A|||_\infty = \max(5,2,2) = 5$.

b) We use the tableau method to estimate A^{-1}:

$$\left[\begin{array}{ccc|ccc} -2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & -2 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right]$$

Thus, $A^{-1} = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & -1 \\ 1 & 2 & -2 \end{bmatrix}$. We deduce $|||A^{-1}|||_1 = 7$, $|||A^{-1}|||_\infty = 6$

c) Since $|||.|||$ is a matrix norm, we have:

$$|||A^{-1} \cdot A||| \leq |||A||| \cdot |||A^{-1}|||$$

Moreover $|||A^{-1} \cdot A||| = |||\mathrm{Id}||| = 1$. Indeed:

$$|||\mathrm{Id}||| = \max_{\|x\|=1} \|\mathrm{Id}x\| = \max_{\|x\|=1} \|x\| = 1.$$

Therefore, $1 \leq |||A||| \cdot |||A^{-1}|||$ and therefore:

$$\frac{1}{|||A|||} \leq |||A^{-1}|||.$$

Exercise 8.

- The Jacobi method gives the following matrix:

$$J_j = D^{-1}(L+U) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/3 & 0 \end{bmatrix}.$$

  Unfortunately, compute the norms $|||.|||_1$ or $|||.|||_\infty$ are inconclusive to determine the Jacobi method converges ($|||J_j|||_1 = |||J_j|||_\infty = 1$). Thus, we estimate the spectral radius of J_j by estimating its eigenvalues:

$$P_{J_j}(\lambda) = \lambda^2 - \mathrm{trace} J_j \cdot \lambda + \det(J_j) = \lambda^2 - 0 \cdot \lambda - \frac{1}{3}.$$

  Thus, the eigenvalues are $\lambda = \pm\sqrt{\frac{1}{3}}$ and $\rho(J_j) = \sqrt{\frac{1}{3}} < 1$. Therefore, the Jacobi method converges.

- The Gauss-Seidel method gives the following matrix:

$$J_g = (D-L)^{-1}U = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

  The eigenvalues of J_g are $\lambda = 0, \frac{1}{3}$. Thus, $\rho(J_g) = \frac{1}{3} < 1$. The method converges.

- The SOR method gives:

$$\begin{aligned} J_\omega &= (D-\omega L)^{-1}\Big((1-\omega)D + \omega U\Big) = \begin{bmatrix} 2 & 0 \\ -\omega & 3 \end{bmatrix}^{-1} \begin{bmatrix} 2(1-\omega) & 2\omega \\ 0 & 3(1-\omega) \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 3 & 0 \\ \omega & 2 \end{bmatrix} \begin{bmatrix} 2(1-\omega) & 2\omega \\ 0 & 3(1-\omega) \end{bmatrix} = \begin{bmatrix} (1-\omega) & \omega \\ \frac{1}{3}\cdot\omega(1-\omega) & \frac{\omega^2}{3} + (1-\omega) \end{bmatrix}. \end{aligned}$$

  The characteristic polynomial of J_ω is given by:

$$\begin{aligned} P_{J_\omega}(\lambda) &= \lambda^2 - \frac{\omega^2}{3} \cdot \lambda + (1-\omega)\Big(\omega^2/3 + (1-\omega)\Big) - \omega^2/3 \cdot (1-\omega) \\ &= \lambda^2 - \frac{\omega^2}{3} \cdot \lambda + (1-\omega)^2. \end{aligned}$$

  Thus, the eigenvalues are $\lambda = \frac{1}{2} \cdot \Big(\frac{\omega^2}{3} \pm \sqrt{\frac{\omega^4}{9} - 4(1-\omega)^2}\Big)$.

Exercise 9.

- Jacobi method:

$$J_j = D^{-1}(L+U) = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of J_j is:

$$P_{J_j}(\lambda) = \begin{vmatrix} -\lambda & 1/3 & 1/3 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1/3 \\ 1 & -\lambda \end{vmatrix} = -\lambda(\lambda^2 - 1/3).$$

Thus, $\lambda = 0, \pm\sqrt{1/3}$. Since $\rho(J_j) = \sqrt{1/3} < 1$, the Jacobi method converges.

- Gauss-Seidel method:

$$J_g = (D-L)^{-1}U = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1 & 0 \\ 1/3 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 1/3 & 1/3 \end{bmatrix}.$$

Thus, $|||J_g|||_\infty = 2/3 < 1$. Therefore, the Gauss-Seidel converges