MAT 423: Homework 10 (11/26)

Ex 1.

Consider the following minimization problem: minimize $J(x_1, x_2, x_3) = x_1 + 2x_2 + 2x_3$ over the domain Ω:

$$\begin{aligned} \mathbf{x} &\geq 0, \\ x_1 + 2x_2 + 4x_3 &\geq 1 \\ -4x_1 - 2x_2 - x_3 &\geq -3 \end{aligned}$$

a) Let $\omega_1 = x_1 + 2x_2 + 4x_3 - 1$, $\omega_2 = -4x_1 - 2x_2 - x_3 + 3$. We look for a corner of the form $\tilde{\mathbf{x}}_c = (*, 0, 0, *, 0)$ which leads to:

$$\begin{aligned} x_1 - \omega_1 &= 1 \\ -4x_1 &= -3 \end{aligned}$$

Thus, $x_1 = 3/4$ and $\omega_1 = -1/4$. Since $\omega_1 < 0$, this corner is not feasible. We try now $\tilde{\mathbf{x}} = (0, *, 0, *, 0)$ leading to:

$$\begin{aligned} 2x_2 - \omega_1 &= 1 \\ -2x_2 &= -3 \end{aligned}$$

which gives $x_2 = 3/2$ and $\omega_1 = 2$. Thus, the corner $\mathbf{x}_c =$ is feasible.

To find the minimizer, we introduce the tableau:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 4 & -1 & 0 & 1 \\ -4 & -2 & -1 & 0 & -1 & -3 \\ \hline 1 & 2 & 3 & 0 & 0 & 0 \end{array}\right].$$

We reduce using the 2nd and 4th column as "hot vectors":

$$\left[\begin{array}{ccccc|c} 3 & 0 & -3 & 1 & 1 & 2 \\ 2 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{3}{2} \\ \hline -3 & 0 & 1 & 0 & -1 & -3 \end{array}\right].$$

Then we use x_1 as a new entry variable and reduce the tableau to have a hot vector in the first column:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{5}{2} & -\frac{2}{3} & -\frac{1}{6} & \frac{1}{6} \\ \hline 0 & 0 & -2 & 1 & 0 & -1 \end{array}\right].$$

The corner is still not optimal, we can use the third column as hot vector and reduce:

$$\left[\begin{array}{ccccc|c} 1 & \frac{2}{5} & 0 & \frac{1}{15} & \frac{4}{15} & \frac{11}{15} \\ 0 & \frac{2}{5} & 1 & -\frac{4}{15} & -\frac{1}{15} & \frac{1}{15} \\ \hline 0 & \frac{4}{5} & 0 & \frac{7}{15} & -\frac{2}{15} & -\frac{13}{15} \end{array}\right].$$

Finally, using the fifth column as hot vector leads to:

$$\left[\begin{array}{ccccc|c} \frac{15}{4} & \frac{3}{2} & 0 & \frac{1}{4} & 1 & \frac{11}{4} \\ \frac{1}{4} & \frac{1}{2} & 1 & -\frac{1}{4} & 0 & \frac{1}{4} \\ \hline \frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array}\right].$$

Thus the optimal corner is at $\tilde{\mathbf{x}}_* = (0, 0, \frac{1}{4}, 0, \frac{11}{4})$ where the cost is $J(\tilde{\mathbf{x}}_*) = \frac{1}{2}$.

b) The dual problem is to *maximize* $G(y) = y_1 - 3y_2$ under the constraints:

$$A^T\mathbf{y} \leq \mathbf{c} \quad \Rightarrow \quad \begin{cases} y_1 - 4y_2 & \leq & 1 \\ 2y_1 - 2y_2 & \leq & 2 \\ 4y_1 - y_2 & \leq & 2 \end{cases}$$

We draw the feasible set for the dual problem in figure 1. There are two corners and the maximum is reached at $\mathbf{y}_* = (\frac{1}{2}, 0)$ where $G(\mathbf{y}_*) = \frac{1}{2}$ $(= J(\mathbf{x}_*))$.

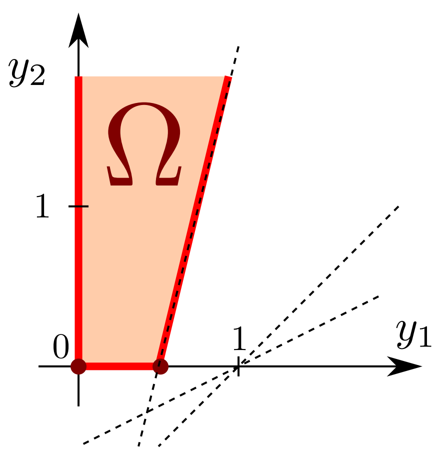

Figure 1: Feasible set for the dual problem. There are two feasible corners at $(0, 0)$ and $(\frac{1}{2}, 0)$.

Ex 2.

Consider the minimization problem $J(\mathbf{x}) = \langle \mathbf{c}, \mathbf{x} \rangle$ with $\mathbf{c} = (1, 1, 1, 3)^T$ with the constraint

$$A\mathbf{x} \geq \mathbf{b} \qquad \text{with} \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = (1, 1, 1, 1)^T.$$

a) We have $A\bar{\mathbf{x}} = (1, 2, 4, 2)^T \geq \mathbf{b}$ and $A^T\bar{\mathbf{y}} = (1, 1, 1, 1) \leq \mathbf{c}$.

b) Moreover $J(\bar{\mathbf{x}}) = 3 = G(\bar{\mathbf{y}}) = 3$, therefore $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ are optimal.

Ex 3.

To show that the primal problem is feasible, we need to show that Ω is non-empty is that $J(\mathbf{x}) = \langle \mathbf{c}, \mathbf{x} \rangle$ is lower-bounded. Since $A > 0$, taking $\mathbf{x}_k = (k, \dots, k)$, we have each component of $A\mathbf{x}_k$ goes to infinity as $k \to +\infty$. Thus, there exists $K > 0$ large enough such that: $A\mathbf{x}_K \geq \mathbf{b}$. Moreover, since $\mathbf{c} \geq 0$ and $\mathbf{x} \geq 0$, we have $J(\mathbf{x}) \geq 0$. Therefore, the primal problem is feasible.

For the dual problem, the feasible set is non-empty since $A^T\mathbf{0} = \mathbf{0} \leq \mathbf{c}$ since $\mathbf{c} \geq 0$. Moreover, if $\mathbf{y} = (y_1, \dots, y_m)$ in the feasible set, each coordinate y_j is bounded (otherwise $A^T\mathbf{y}$ goes to infinity and cannot satisfy $A^T\mathbf{y} \leq \mathbf{c}$). Therefore $G(\mathbf{y})$ upper-bounded. Therefore, the dual problem is also feasible.

Ex 4.

Here A is fixed, thus the feasible set Ω is unchanged and the (finite number of) corners stay the same $\mathbf{x}_1, \dots, \mathbf{x}_k$. For a given $\mathbf{c}$, the unique optimal $\mathbf{x}_*$ has to be one of the corner, let's denote it i_*. Since it is unique:

$$J_{\mathbf{c}}(\mathbf{x}_{i_*}) > \max_{i \neq i_*} J_{\mathbf{c}}(\mathbf{x}_i).$$

By continuity, if we change $\mathbf{c}$ a little (denoted $\tilde{\mathbf{c}}$) then:

$$J_{\mathbf{c}}(\mathbf{x}_i) \approx J_{\tilde{\mathbf{c}}}(\mathbf{x}_i).$$

Therefore, we will still have:

$$J_{\tilde{\mathbf{c}}}(\mathbf{x}_{i_*}) > \max_{i \neq i_*} J_{\tilde{\mathbf{c}}}(\mathbf{x}_i).$$

Therefore, $\mathbf{x}_{i_*}$ will be still the optimal.