MAT 423: Homework 7 (10/29)

Ex 1. [4pts]

Consider $\Omega = [-1,1]\times[-1,1]$. Let's first show that $G(\Omega) \subset \Omega$. Take $-1 \le x, y \le 1$:

$$-\frac{1}{3} \;\le\; \frac{y}{3}\cos x \;\le\; \frac{1}{3} \qquad \text{and} \qquad 0 \;\le\; \frac{1}{4}\mathrm{e}^{-x^2-y^2} \;\le\; \frac{1}{4}.$$ 1pt

Thus, $G(x,y) \in \Omega$.

Then we estimate the Jacobian of G:

$$DG(x,y) = \begin{bmatrix} -\frac{y}{3}\sin x & \frac{1}{3}\cos x \\ -\frac{x}{2}\mathrm{e}^{-x^2-y^2} & -\frac{y}{2}\mathrm{e}^{-x^2-y^2} \end{bmatrix}$$ 1pt

The norm $\|.\|_1$ of $DG(x,y)$ on Ω is bounded by:

$$\max_{(x,y)\in\Omega} |||DG(x,y)|||_1 \le \max(\frac{1}{3}+\frac{1}{2}, \frac{1}{3}+\frac{1}{2}) = \frac{5}{6}.$$ 1pt

Thus, for any $\mathbf{x},\mathbf{y} \in \Omega$, $\|G(\mathbf{x}) - G(\mathbf{y})\|_1 \le \kappa\|\mathbf{x}-\mathbf{y}\|_1$ with $\kappa = \frac{5}{6}$. We conclude that G is a contraction on Ω.

We conclude that G has a unique fixed point on Ω. 1pt

Ex 2.

Consider the function $G(x,y) = \begin{pmatrix} \frac{1}{2}\Big(\ln(1+x^2)\; + y\Big) \\ \frac{1}{10}y(x^2+y^2) \end{pmatrix}$. We notice that $(0,0)$ is a fixed point of G. To show that the iterative sequence (x_k, y_k) converges to $(0,0)$, we can show that G is a contraction on a domain Ω including (x_0, y_0). Then, the sequence will have to converge to the unique fixed point, i.e. $(0,0)$. Let's show that G is a contraction on $\Omega = [0,1]\times[0,1]$.

Take $(x,y) \in \Omega$, we have:

$$0 \;\le\; \frac{1}{2}\Big(\ln(1+x^2)\; + y\Big) \;\le\; \frac{1}{2}(\ln 2 + 1) \qquad \text{and} \qquad 0 \;\le\; \frac{1}{10}y(x^2+y^2) \;\le\; \frac{3}{10}.$$

Thus, $G(\Omega) \subset \Omega$.

The Jacobian of G is given by

$$DG(x,y) = \begin{bmatrix} \frac{x}{1+x^2} & \frac{y}{2} \\ \frac{2xy}{10} & \frac{3y^2}{10} \end{bmatrix}.$$

There is an additional difficulty to upper-bound the term $\frac{x}{1+x^2}$ (the upper bound 1 is not tight enough). But notice that $1+x^2 \geq 2x$ (since $(1-x)^2 \geq 0$), therefore $1 \geq \frac{2x}{1+x^2}$. Thus, the norm $\|.\|_1$ of $DG(x,y)$ on Ω is bounded by:

$$\max_{(x,y)\in\Omega} |||DG(x,y)|||_1 \leq \max(\frac{1}{2}+\frac{2}{10}, \frac{1}{2}+\frac{3}{10}) = \frac{8}{10}.$$

Thus, for any $\mathbf{x},\mathbf{y} \in \Omega$, $\|G(\mathbf{x})-G(\mathbf{y})\|_1 \leq \kappa\|\mathbf{x}-\mathbf{y}\|_1$ with $\kappa = \frac{4}{5}$. We conclude that G is a contraction on Ω.

Ex 3. [3pts]

a) Solving $F(x,y) = (0,0)$ gives:

$$y(3x^2-y^2) = 0 \Rightarrow y = 0 \quad \text{or} \quad y = \pm\sqrt{3}x.$$

Taking $y = 0$, we then solve $x^3+1 = 0$ which gives $x = -1$. Thus a solution to $F(x,y) = (0,0)$ is given by $(x_*, y_*) = (-1, 0)$.

1pt

b) The Jacobian of F is defined as

$$DF(x,y) = \begin{bmatrix} 3x^2-3y^2 & -6xy \\ 6xy & -3y^2 \end{bmatrix}$$ 1pt

Starting at $(x_0,y_0) = (-0.8, 0.2)$, the Newton method gives:

$$\mathbf{x}^{(1)} = \begin{pmatrix} -.8 \\ .2 \end{pmatrix} - \begin{bmatrix} 1.8 & 0.96 \\ -0.96 & 1.8 \end{bmatrix}^{-1} \begin{pmatrix} .584 \\ .376 \end{pmatrix} = \begin{pmatrix} -0.966.. \\ -0.097.. \end{pmatrix}.$$ 1pt

Extra) Perform a linear regression of the form:

$$\ln\|F(\mathbf{x}^{(k+1)})\| \quad \approx \quad \alpha \ln\|F(\mathbf{x}^{(k)})\| + \beta,$$ +1pt

gives $\alpha = 2.0018$ which is consistent with the Newton's method converging quadratically.

Ex 4. [3pts]

Let $J(x,y) = 2x^2+y^2+(x^2+1)\sin 4y$. We have:

2pts

$$\nabla J(x,y) = \begin{pmatrix} 4x+2x\sin 4y \\ 2y+4(x^2+1)\cos 4y \end{pmatrix} \quad , \quad D^2J(x,y) = \begin{bmatrix} 4+2\sin 4y & 8x\cos 4y \\ 8x\cos 4y & 2-16(x^2+1)\sin 4y \end{bmatrix}$$

Thus,

$$\mathbf{x}_1 = \mathbf{x}_0 - \left[D^2J(\mathbf{x}_0)\right]^{-1}\nabla J(\mathbf{x}_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$ 1pt

Ex 5.

1pt

The code is given below. The convergence of the method is given in figure 1. The rate of convergence gives $\alpha \approx 1.538$ which not as fast as the Newton's method but faster than a geometric convergence.

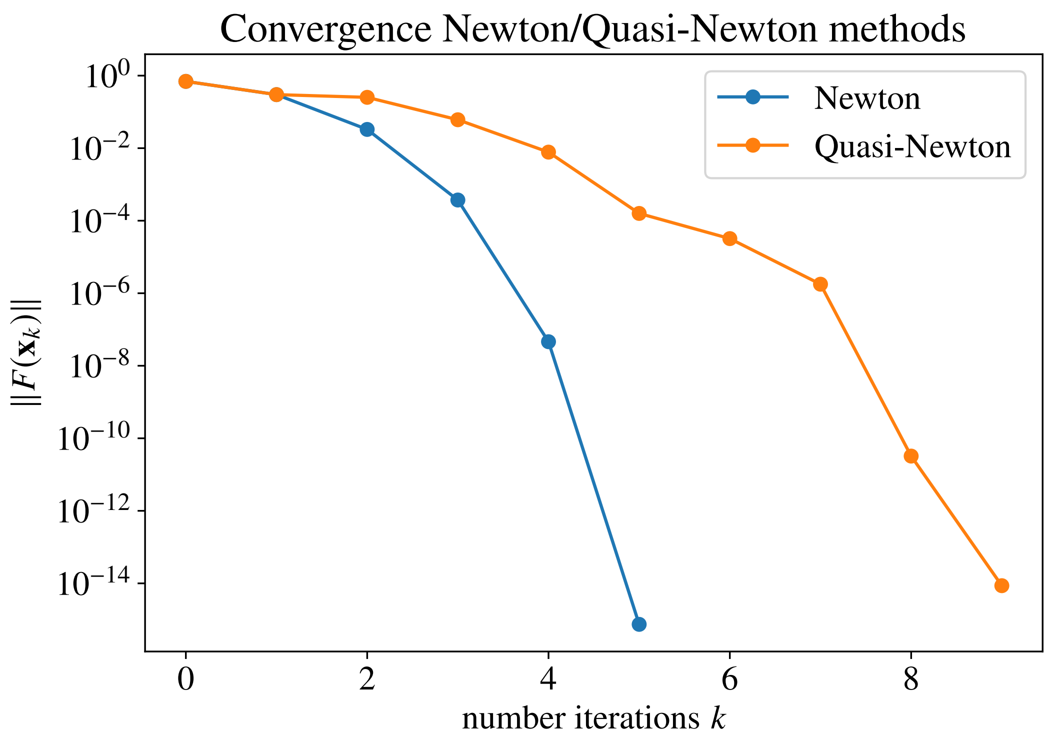

Figure 1: Comparison of the Newton and quasi-Newton methods applied to find a zero of $F(x, y)$.

```
#=
  Implementation of a quasi-Newton method (Broyen)
=#
using LinearAlgebra
# Parameters
N = 5
F(x,y) = [x^3-3*x*y^2+1; 3*x^2*y-y^3]
DF(x,y) = [3*x^2-3*y^2 -6*x*y;
           6*x*y 3*x^2-3*y^2]
X0 = [-.8; 0.2]
N = 9    # number of steps
# init
norm_seqFX_Broyden = zeros(N+1)
norm_seqFX_Broyden[1] = norm(F(Xprev[1],Xprev[2]))
norm_seqFX_Broyden[2] = norm(F(X[1],X[2]))
# first step (Newton's method)
Xprev = copy(X0)
A = DF(Xprev[1],Xprev[2])
invA = inv(A)
X = Xprev - invA*F(Xprev[1],Xprev[2])
# loop
for k=2:N
    # update A
    dx, dF = X-Xprev, F(X[1],X[2])-F(Xprev[1],Xprev[2])
    u = (dF - A*dx)/norm(dx)^2
    invA .+= - invA*(u*dx')*invA/(1+dot((invA*u),dx))
    # testing
    #A .+= u*dx'
    #println(A*invA)
    # update X
    Xnew = X - invA*F(X[1],X[2])
    Xprev .= X
    X .= Xnew
    norm_seqFX_Broyden[k+1] = norm(F(X[1],X[2]))
end
```