MAT 423: Practice final exam

Exercise 1.

Consider the function $g(x) = \frac{1}{2} + \frac{1}{2}\ln(1+x^2)$.

a) Show that g is a contraction mapping on $\mathbb{R}$.

b) Propose an algorithm to find the fixed point x_*.
Estimate the number of iterations needed to have an accuracy of 10^{-4}.

Exercise 2.

Consider the function $G(x,y) = \Big(\frac{1}{3}\big(x\ln(1+y)+y\big),\ \frac{1}{3}\big(\mathrm{atan}(1+x)+1\big)\Big)$.

a) Show that G is a contraction mapping on $\Omega = [0,1]\times[0,1]$.

b) Propose an algorithm to find the fixed point (x_*, y_*).
Estimate the number of iterations needed to have an accuracy of 10^{-4}.

Exercise 3.

Consider the function $f(x) = \mathrm{e}^{2x} - 2\cos x$

a) Show that there exists $x_* \in \mathbb{R}$ such $f(x_*) = 0$.

b) Using the bisection method, how many iterations would you need to find a zero x_* with accuracy 10^{-4}?

c) Starting with $x_0 = 0$, apply the Newton's method once to f.

Exercise 4.

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}.$$

a) Let $\mathbf{b} = (1,1,1)$. Find $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

b) Find the LU decomposition of A.

Exercise 5.

a) Suppose A symmetric matrix with two eigenvalues/eigenvectors $(\lambda_1, \mathbf{u}_1)$ and $(\lambda_2, \mathbf{u}_2)$ (i.e. $A\mathbf{u}_1 = \lambda_1\mathbf{u}_1$). Show that if $\lambda_1 \neq \lambda_2$ then $\langle \mathbf{u}_1,\ \mathbf{u}_2\rangle = 0$.
Hint: compare $\langle A\mathbf{u}_1, \mathbf{u}_2\rangle$ *and* $\langle \mathbf{u}_1, A\mathbf{u}_2\rangle$.

b) Take $\mathbf{u} \in \mathbb{R}^n$ and $A = u \otimes u$ (i.e. $A = \mathbf{u}\mathbf{u}^T$). If $n > 1$, show that $\det(A) = 0$.
Hint: consider $\mathbf{v} \perp \mathbf{u}$ *and compute* $A\mathbf{v}$.

Exercise 6.

a) Find a matrix A such that: $|||A|||_1 \geq 2|||A|||_\infty$.

b) Find a matrix B such that: $|||B|||_\infty \geq 3|||B|||_1$.

Exercise 7.

Consider the matrix: $A = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}$. Discuss whether the Jacobi and Gauss-Seidel methods could be applied to A.

Extra. Compute the eigenvalues of T_ω the matrix associated with the SOR method.

Exercise 8.

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and $\mathbf{b} = (1, 0, 0)$. The goal is to use the conjugate gradient method to solve $A\mathbf{x} = \mathbf{b}$.

a) Starting from the canonical basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, construct a set of vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ satisfying:

$$\langle A\mathbf{u}_i, \mathbf{u}_j \rangle = 0 \qquad \text{for any } i \neq j.$$

Hint: use the Gram-Schmidt decomposition

b) Take $\mathbf{x}_0 = \mathbf{b}$. Use the vectors $\{\mathbf{u}_i\}_i$ to implement the conjugate gradient method:

$$\mathbf{x}_k = \mathbf{x}_{k-1} - \langle A\mathbf{x}_{k-1} - \mathbf{b}, \mathbf{u}_k \rangle \frac{\mathbf{u}_k}{\langle A\mathbf{u}_k, \mathbf{u}_k \rangle} \qquad k = 1 \ldots 3$$

Exercise 9.

Consider the function $J(x, y) = (x^2 - 1)^2 + y^2$.

a) Apply the Newton's method once to ∇J starting at $\mathbf{x}_0 = (\frac{1}{2}, 0)$.

b) Apply the gradient descent once to J starting at $\mathbf{x}_0 = (\frac{1}{2}, 0)$ with the learning rate $\lambda = .1$.

c) Starting at $\mathbf{x}_0 = (\varepsilon, 0)$ with $0 < \varepsilon \ll 1$, discuss the behavior of the two sequences $\{\mathbf{x}_n\}_n$ given by the Newton's method and gradient descent.

Exercise 10.

Find the minimum of the function $J(x_1, x_2) = x_1 - 2x_2$ over the constraints

$$x_1, x_2 \geq 0 \quad |x_1 - 1| + |x_2 - 1| \leq 1.$$

Exercise 11.

Consider the *primal* problem of minimizing $J(x_1, x_2, x_3) = x_1 + 2x_2 + x_3$ under the constraints:

$$\mathbf{x} \geq 0 \quad , \quad A\mathbf{x} \geq \mathbf{b} \quad \text{with } A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = (1, 4)^T.$$

a) Find a corner of the form $\tilde{\mathbf{x}}_c = (0, *, 0, *, 0)$.

b) Apply the simplex method to find the minimum.

c) Write down the dual problem and find the maximum.

Exercise 12.

Consider the constraints $x_1, x_2 \geq 0$ and

$$\begin{cases} x_1 + x_2 & \geq & 1 \\ -x_1 + x_2 & \geq & -1 \end{cases}$$

Find a vector $\mathbf{c}$ such that the problem of minimzing $J(\mathbf{x}) = \langle \mathbf{c}, \mathbf{x} \rangle$ over the constraints has:

a) no solution (i.e. J not lower bounded over the feasible domain)

b) multiple solutions

c) a unique solution