MAT 423: Practice final exam (solution)

Exercise 1.

Let $g(x) = \frac{1}{2} + \frac{1}{2}\ln(1+x^2)$.

a) First, $g(\mathbb{R}) \subset \mathbb{R}$. Second, $g'(x) = \frac{1}{2}\frac{2x}{1+x^2}$. Since $1+x^2 \geq |2x|$ (since $1+x^2-|2x| \geq (1-|x|)^2 \geq 0$), we have:

$$\max_{x\in\mathbb{R}} |g'(x)| \leq \frac{1}{2}.$$

Therefore, we deduce that $|g(x)-g(y)| \leq \frac{1}{2}|x-y|$. We conclude that g is contraction mapping on $\mathbb{R}$.

b) To find the fixed point x_* of g, we simply use the fixed-point method: take $x_0 = 0 \in \mathbb{R}$ and iterate

$$x_{n+1} = g(x_n)i$$

To obtain an accuracy of 10^{-4}, a sufficient condition is to have n satisfies ($k = \frac{1}{2}$):

$$\frac{k^n}{1-k}|x_1 - x_0| \leq 10^{-4} \quad \Rightarrow \quad \frac{\frac{1}{2}^n}{\frac{1}{2}}\left|\frac{1}{2} - 0\right| \leq 10^{-4} \quad \Rightarrow \quad n \geq -4\frac{\ln 10}{\ln\frac{1}{2}} \approx 13.29$$

Therefore, $n \geq 14$ is a sufficient condition.

Exercise 2.

Consider the function $G(x,y) = \left(\frac{1}{3}\Big(x\ln(1+y)+y\Big), \frac{1}{3}\Big(\text{atan}(1+x)+1\Big)\right)$.

a) Taking $0 \leq x, y \leq 1$, we find:

$$0 \leq \frac{1}{3}\Big(x\ln(1+y)+y\Big) \leq \frac{1}{3}(\ln 2 + 1) \leq 1$$
$$0 \leq \frac{1}{3}\Big(\text{atan}(1+x) \leq \frac{1}{3}\frac{\pi}{2} \leq 1$$

Thus, $G(x,y) \in [0,1]\times[0,1]$. Moreover,

$$DG(x,y) = \frac{1}{3}\begin{bmatrix} \ln(1+y) & (\frac{x}{1+y}+1) \\ \frac{1}{1+(1+x)^2} & 0 \end{bmatrix}$$

Taking the norm $\|.\|_1$, we find that on the domain $[0,1]\times[0,1]$:

$$\||DG(x,y)\||_1 \leq \frac{1}{3}\max(\ln 2 + 1, 2+0) = \frac{2}{3}.$$

Therefore, for any $\mathbf{x}$ and $\mathbf{y}$ in $[0,1]\times[0,1]$:

$$\|G(\mathbf{x}) - G(\mathbf{y})\|_1 \leq \frac{2}{3}\|\mathbf{x}-\mathbf{y}\|_1.$$

Thus, G contraction on $[0,1]\times[0,1]$.

b) To find the fixed point of G on $[0,1]\times[0,1]$, we can take for example $\mathbf{x}_0 = (0,0)$ and iterate:

$$\mathbf{x}_{n+1} = G(\mathbf{x}_n).$$

To have accuracy 10^{-4}, a sufficient condition is to have n satisfying:

$$\frac{\kappa^n}{1-\kappa}\|\mathbf{x}_0 - \mathbf{x}_1\|_1 \leq 10^{-4}.$$

Exercise 3.

Consider the function $f(x) = \mathrm{e}^{2x} - 2\cos x$

a) Take $a = 0$, $f(a) = 1 - 2 = -1 < 0$. Moreover, taking $b = 1$, we obtain $f(b) = \mathrm{e}^1 - 2\cdot\cos 1 >$
$expo - 2 > 0$. Since f is a continuous function, there exists $x_* \in [a,b]$ satisfying $f(x_*) = 0$.

b) A sufficient condition to have accuracy 10^{-4} is to have n satisfying:

$$\frac{1}{2^{n+1}}|1-0| \leq 10^{-4} \quad \Rightarrow \quad (n+1)\ln\frac{1}{2} \leq -4\ln 10 \quad \Rightarrow \quad n \geq -4\frac{\ln 10}{\ln\frac{1}{2}} - 1 \approx 12.29.$$

Thus, $n \geq 13$ is enough.

c) Newton's method gives:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1-2}{2-2\cdot 0} = \frac{1}{2}.$$

Exercise 4.

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}.$$

a) We use the tableau to $\mathbf{x}$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 6 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & -2 & -3 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus, $x_3 = 0$. We deduce $x_2 = 1$ and finally $x_1 = -1$.

b) Let's compute the LU decomposition:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 3 & 4 & 6 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & -2 & -3 & -3 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

Thus, $MA = U$ with

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

Then, we estimate the inverse of M:

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{array}\right]$$

Thus, $L = M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and $A = LU$.

Exercise 5.

a) We have $\langle A\mathbf{u}_1,\, \mathbf{u}_2\rangle = \langle \lambda_1\mathbf{u}_1,\, \mathbf{u}_2\rangle$. Moreover,

$$\langle A\mathbf{u}_1,\, \mathbf{u}_2\rangle = \langle \mathbf{u}_1,\, A\mathbf{u}_2\rangle = \langle \mathbf{u}_1,\, \lambda_2\mathbf{u}_2\rangle.$$

Therefore, $\lambda_1\langle \mathbf{u}_1,\, \mathbf{u}_2\rangle = \lambda_2\langle \mathbf{u}_1,\, \mathbf{u}_2\rangle$. Since $\lambda_1 \neq \lambda_2$, we must have $\langle \mathbf{u}_1,\, \mathbf{u}_2\rangle = 0$.

b) Since $n > 1$, there exists $\mathbf{v} \neq 0$ such that $\mathbf{v} \perp \mathbf{u}$. We have:

$$A\mathbf{v} = \mathbf{u} \otimes \mathbf{u}\mathbf{v} = \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u} = 0.$$

Therefore, A is singular and $\det(A) = 0$.

Exercise 6.

a) Take $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. We have $|||A|||_1 = 2$ and $|||A|||_\infty = 1$.

b) Take $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. We have $|||B|||_\infty = 3$ and $|||B|||_1 = 1$.

Exercise 7.

- Jacobi: $J_j = D^{-1}(L+U) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ -1 & 0 \end{bmatrix}$. Since $|||J_j|||_\infty = |||J_j|||_1 = 1$, we cannot directly deduce that J_j is a contraction. We investigate the eigenvalues of J_j:

$$P(\lambda) = \lambda^2 + \frac{1}{3}$$

thus $\lambda = \pm i\sqrt{\frac{1}{3}}$. Therefore the spectral radius is given by $\rho(J_j) = \sqrt{\frac{1}{3}} < 1$ and thus the Jacobi converges.

- Gauss-Seidel:

$$J_g = (D-L)^{-1}U = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/3 \\ 0 & -1/3 \end{bmatrix}.$$

Since $|||J_g|||_1 = \frac{1}{3} < 1$, Gauss-Seidel method converges.

- SOR:

$$\begin{aligned} J_\omega &= (D-\omega L)^{-1}\Big((1-\omega)D+\omega U\Big) = \begin{bmatrix} 3 & 0 \\ 2\omega & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix} \\ &= \frac{1}{6}\begin{bmatrix} 2 & 0 \\ -2\omega & 3 \end{bmatrix}\begin{bmatrix} 3(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix} = \begin{bmatrix} (1-\omega) & \omega/3 \\ -\omega(1-\omega) & -\omega^2/3+(1-\omega) \end{bmatrix} \end{aligned}$$

The characteristic polynomial is given by:

$$P(\lambda) = \lambda^2 - (-\omega^2/3 + 2(1-\omega))\cdot\lambda + (1-\omega)^2.$$

Thus, the eigenvalues are:

$$\lambda = \frac{1}{2}\left(-\omega^2/3 + 2(1-\omega) \pm \sqrt{(-\omega^2/3+2(1-\omega))^2 - 4(1-\omega)^2}\right).$$

Exercise 8.

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and $\mathbf{b} = (1,0,0)$. The goal is to use the conjugate gradient method to solve $A\mathbf{x} = \mathbf{b}$.

a) We consider $\mathbf{u}_1 = \mathbf{e}_1 = (1,0,0)^T$. Then:

$$\mathbf{u}_2 = \mathbf{e}_2 - \frac{\langle A\mathbf{u}_1, \mathbf{e}_2\rangle}{\langle A\mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 = \mathbf{e}_2 - (-\frac{1}{2})\mathbf{u}_1 = (\frac{1}{2}, 1, 0)^T.$$

and

$$\mathbf{u}_3 = \mathbf{e}_3 - \frac{\langle A\mathbf{u}_1, \mathbf{e}_3\rangle}{\langle A\mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle A\mathbf{u}_2, \mathbf{e}_3\rangle}{\langle A\mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 = (1/3, 2/3, 1)^T.$$

b) Take $\mathbf{x}_0 = \mathbf{b}$. We obtain:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0 - \langle A\mathbf{x}_0 - \mathbf{b}, \mathbf{u}_1\rangle \frac{\mathbf{u}_1}{\langle A\mathbf{u}_1, \mathbf{u}_1\rangle} = (\frac{1}{2}, 0, 0)^T \\ \mathbf{x}_2 &= \mathbf{x}_1 - \langle A\mathbf{x}_1 - \mathbf{b}, \mathbf{u}_2\rangle \frac{\mathbf{u}_2}{\langle A\mathbf{u}_2, \mathbf{u}_2\rangle} = (\frac{2}{3}, \frac{1}{3}, 0)^T \\ \mathbf{x}_3 &= \mathbf{x}_2 - \langle A\mathbf{x}_2 - \mathbf{b}, \mathbf{u}_3\rangle \frac{\mathbf{u}_3}{\langle A\mathbf{u}_3, \mathbf{u}_3\rangle} = (\frac{3}{4}, \frac{1}{2}, \frac{1}{4})^T \end{aligned}$$

and we find $A\mathbf{x}_3 = \mathbf{b}$.

Exercise 9.

Consider the function $J(x,y) = (x^2-1)^2 + y^2$. We have:

$$\nabla J(x,y) = \begin{pmatrix} 4x(x^2-1) \\ 2y \end{pmatrix} \quad \text{and} \quad D^2 J(x,y) = \begin{bmatrix} 4(x^2-1)+8x^2 & 0 \\ 0 & 2 \end{bmatrix}.$$

a) Newton's method gives:

$$\mathbf{x}_1 = \mathbf{x}_0 - [D^2 J(\mathbf{x}_0)]^{-1}\nabla J(\mathbf{x}_0) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

b) The gradient descent gives:

$$\mathbf{x}_1 = \mathbf{x}_0 - \lambda\nabla J(\mathbf{x}_0) = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} - .1 \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} .65 \\ 0 \end{pmatrix}.$$

c) Starting at $\mathbf{x}_0 = (\varepsilon, 0)$, the Newton's method will converge to the critical point nearby $\mathbf{x}_* = (0,0)$ whereas the gradient descent will move away and converge to the global minimum $(1,0)$.

Exercise 10.

The feasible domain is a square with 4 corners at $(1,0)$, $(0,1)$, $(2,1)$ and $(1,2)$ (see figure 1). The minimum at on these corners is at $\mathbf{x}_c = (1,2)$.

Exercise 11.

Consider the *primal* problem of minimizing $J(x_1,x_2,x_3) = x_1 + 2x_2 + x_3$ under the constraints:

$$\mathbf{x} \geq 0 \quad , \quad A\mathbf{x} \geq \mathbf{b} \quad \text{with } A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = (1,4)^T.$$

a) Introducing the tableau:

$$T = \left[\begin{array}{ccccc|c} 1 & 1 & 2 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 & -1 & 4 \\ \hline 1 & 2 & 1 & 0 & 0 & 0 \end{array}\right]$$

Reducing using the 2nd and 4th column gives:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & -1 & 0 & 1 \\ -2 & 0 & -1 & 1 & -1 & 3 \\ \hline -1 & 0 & -3 & 2 & 0 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & -1 & 4 \\ -2 & 0 & -1 & 1 & -1 & 3 \\ \hline 3 & 0 & -1 & 0 & 2 & -8 \end{array}\right]$$

The corner is $\mathbf{x}_c = (0,4,0,3,0)$.

b) The previous corner is not optimal since the law raw is not positive. Taking the 3rd column as basic variable gives:

$$\left[\begin{array}{ccccc|c} -1 & 1 & 1 & 0 & -1 & 4 \\ -3 & 1 & 0 & 1 & -2 & 7 \\ \hline 2 & 1 & 0 & 0 & 1 & -4 \end{array}\right]$$

Thus, we obtain the new corner $\mathbf{x}_c = (0,0,4,7,0)$. This corner is optimal. The minimal cost is 4.

c) The dual problem is written as: finding $\mathbf{y} = (y_1, y_2)^T$ that maximizes $\langle \mathbf{b}, \mathbf{y} \rangle$ under the constraints $\mathbf{y} \geq 0$ and $A^T \mathbf{y} \leq \mathbf{c}$:

$$y_1, y_2 \geq 0 \quad \text{and} \quad \begin{cases} y_1 - y_2 & \leq 1 \\ y_1 + y_2 & \leq 2 \\ 2y_1 + y_2 & \leq 1 \end{cases}$$

Drawing the feasible domain (see fig. 2) shows that there are 3 corners. The maximum is reached at $\mathbf{y}_c = (0, 1)$ where the $\langle \mathbf{b}, \mathbf{y}_c \rangle = 4$ as expected.

Exercise 12.

Consider the constraints $x_1, x_2 \geq 0$ and

$$\begin{cases} x_1 + x_2 & \geq 1 \\ -x_1 + x_2 & \geq -1 \end{cases}$$

a) Taking $\mathbf{c} = (-1, -1)$, the function $J(\mathbf{x}) = \langle \mathbf{c}, \mathbf{x} \rangle$ has no minimum on Ω ($\inf_{\mathbf{x} \in \Omega} J(\mathbf{x}) = -\infty$).

b) Taking $\mathbf{c} = (1, 1)$, both corners $(0, 1)$ and $(1, 0)$ are minimum.

c) With $\mathbf{c} = (2, 1)$, the minimization problem would have a unique minimum at $\mathbf{x}_c = (0, 1)$.

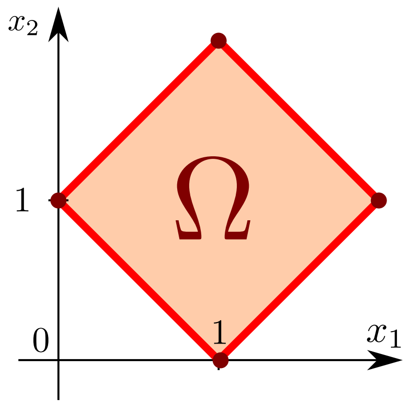

Figure 1: The feasible domain for **Ex. 10**.

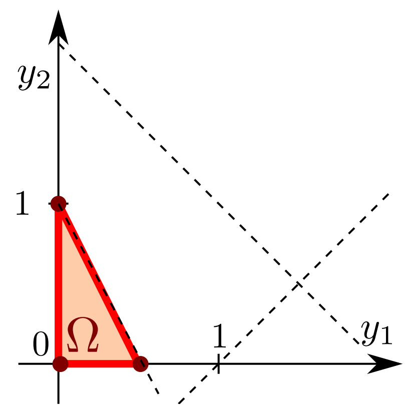

Figure 2: The feasible domain for **Ex. 11**.

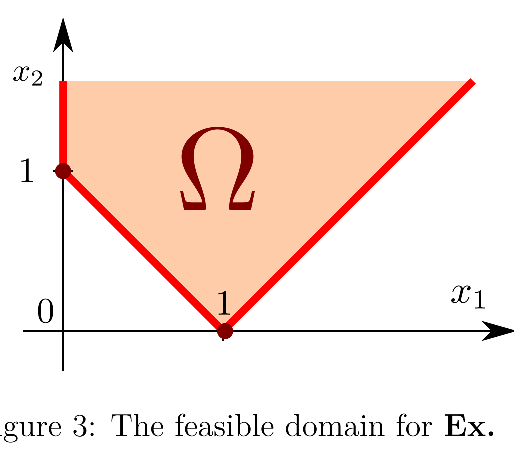

Figure 3: The feasible domain for **Ex. 12**.