MAT 371: Homework 1 (01/15)

1 Chapter 1.2

Ex 3. [4pts]

a) **False**: take the interval $A_n = (0, 1/n)$ in $\mathbb{R}$. We have $A_{n+1} \supset A_n$, but $\cap_n A_n = \varnothing$. 1pt

b) **True**: $\cap_n A_n$ is finite because it is contained in A_1 finite. We can prove that is also non-empty by contradiction. Assume $\cap_n A_n = \varnothing$ and take $x_1 \in A_1$. Since $\cap_n A_n = \varnothing$, there exists n_1 such that $x_1 \notin A_{n_1}$. We can do similarly for all elements $x_1, \ldots, x_k$ of A_1 (there is only a finite number of them) and take $N = \max(n_1, \ldots n_k)$ (no one is left after N). We deduce that A_N does not contain any element x of A_1. Since A_N is also included in A_1 ($A_N \subset A_1$), it is empty. Contradition: all the sets A_n are supposed not empty.

c) **False**: Take A empty and C not empty. 1pt

d) **True**: take $x \in A \cap (B \cap C)$ then 1pt

$$\begin{aligned}
&\Leftrightarrow \quad x \in A \text{ and } x \in B \cap C \\
&\Leftrightarrow \quad x \in A \text{ and } (x \in B \text{ and } x \in C) \\
&\Leftrightarrow \quad x \in A \text{ and } x \in B \text{ and } x \in C \\
&\Leftrightarrow \quad (x \in A \text{ and } x \in B) \text{ and } x \in C \\
&\Leftrightarrow \quad x \in A \cap B \text{ and } x \in C \\
&\Leftrightarrow \quad x \in (A \cap B) \cap C.
\end{aligned}$$

Thus, $A \cap (B \cap C) = (A \cap B) \cap C$.

e) **True**: take $x \in A \cap (B \cup C)$ then 1pt

$$\begin{aligned}
&\Leftrightarrow \quad x \in A \text{ and } x \in B \cup C \\
&\Leftrightarrow \quad x \in A \text{ and } (x \in B \text{ or } x \in C) \\
&\Leftrightarrow \quad (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
&\Leftrightarrow \quad x \in A \cap B \text{ and } x \in A \cap C \\
&\Leftrightarrow \quad x \in (A \cap B) \cup (A \cap C).
\end{aligned}$$

Thus, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Ex 4.

We are going to cut $\mathbb{N}$ in half over and over again. First, take A_0 the even number:

$$A_0 = \{0,\, 2,\, 4,\, \dots\}$$

It remains the odd numbers. We take 'half' of them, in other words:

$$A_1 = \{1,\, 5,\, 9,\, 13, \dots\}.$$

We iterate the process by taking 'half' of the reminding numbers (see figure 1):

$$A_2 = \{3,\, 11,\, 17, \dots\}.$$

It is clear that none of the sets A_n will intersect ($A_i \cap A_j = \varnothing$ if $i \neq j$) and that A_n will always contain an infinite number. The 'trick' is that 'half of infinity' is still infinity.

We can define formally A_n. Take $\alpha_n = 1 + 2 + 4 + \cdots + 2^{n-1} = (2^n - 1)$. Then, for $n \geq 1$, we have

$$A_n = \{\alpha_n,\ \alpha_n + 2^{n+1},\ \alpha_n + 2 \cdot 2^{n+1},\ \alpha_n + 3 \cdot 2^{n+1},\ \alpha_n + 4 \cdot 2^{n+1} \dots\}.$$

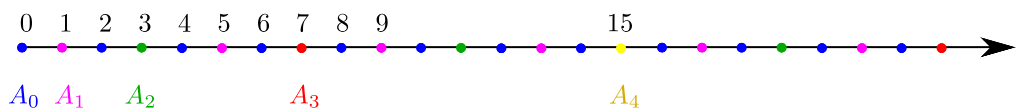

Figure 1: An infinite disjoint union of ensembles A_n that 'paves' $\mathbb{N}$, i.e. $\cup_n A_n = \mathbb{N}$.

Ex 5.

a) Let $x \in (A \cap B)^c$ then:

$$\begin{aligned} &\Rightarrow\ x \notin A \cap B \\ &\Rightarrow\ x \notin A \text{ or } x \notin B \\ &\Rightarrow\ x \in A^c \text{ or } x \in B^c \\ &\Rightarrow\ x \in A^c \cup B^c. \end{aligned}$$

b) Same as a) but going bottom-up.

c) Let's prove the two inclusions at the same time. Let $x \in (A \cup B)^c$ then:

$$\begin{aligned} x \in (A \cup B)^c &\Leftrightarrow\ x \notin A \cup B \\ &\Leftrightarrow\ x \notin A \text{ and } x \notin B \\ &\Leftrightarrow\ x \in A^c \text{ and } x \in B^c \\ &\Leftrightarrow\ x \in A^c \cap B^c. \end{aligned}$$

Ex 7. [4pts]

a) $f(x) = x^2$, $A = [0, 2]$ and $B = [1, 4]$. Notice that f is increasing on both intervals. We have: 1pt

- $f(A) = [0, 4]$, $f(B) = [1, 16]$
- Since $A \cap B = [1, 2]$, $f(A \cap B) = [1, 4]$ which is equal to $f(A) \cap f(B)$. Similarly $f(A \cup B) = f([0, 4]) = [0, 16] = f(A) \cup f(B)$.

b) Take $A = [1, 2]$ and $B = [-2, -1]$. We have $A \cap B = \varnothing$, thus $f(A \cap B) = \varnothing$. However, $f(A) = f(B) = [1, 4]$. Thus, $f(A) \cap f(B) = [1, 4] \neq f(A \cap B)$. 1pt

c) Take $x \in A \cap B$ and $y = g(x)$. We want to show that $y \in g(A) \cap g(B)$. Since $x \in A$, we have $g(x) \in g(A)$. Similarly, $x \in B$ implies that $g(x) \in g(B)$. Thus, $g(x)$ belongs to $g(A)$ and $g(B)$, therefore $g(x) \in g(A) \cap g(B)$. 1pt

d) $g(A \cup B) = g(A) \cup g(B)$. Indeed take $x \in A \cup B$ 1pt

$$\begin{aligned} & x \in A \text{ or } x \in B \\ \Rightarrow \quad & g(x) \in g(A) \text{ or } g(x) \in g(B) \\ \Rightarrow \quad & g(x) \in g(A) \cup g(B). \end{aligned}$$

Thus, $g(A \cup B) \subset g(A) \cup g(B)$. On the other hand: let $y \in g(A) \cup g(B)$. Thus, $y \in g(A)$ or $y \in g(B)$. Therefore there exists x in A or in B such that $y = g(x)$. Since $x \in A \cup B$, we deduce $y \in g(A \cup B)$. Therefore, $g(A) \cup g(B) \subset g(A \cup B)$.

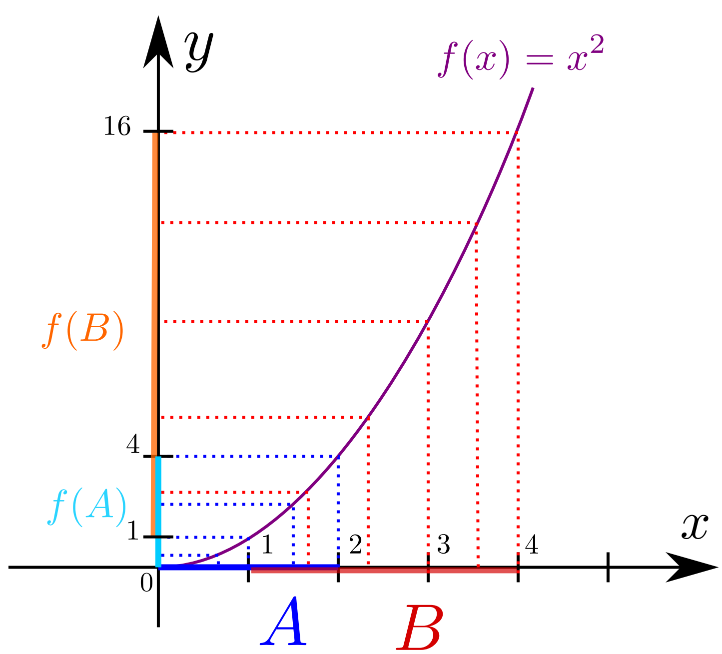

Figure 2: The function $f(x) = x^2$ and the images of $A = [0, 2]$ and $B = [1, 4]$.

Ex 8. [3pts]

a) Take $f(n) = n + 1$. It is injective but not surjective (f does not 'cover' 0). 1pt

b) Take $f(0) = 0$ and $f(n) = n - 1$ for $n \geq 1$. It is surjective but not injective ($f(0) = f(1) = 0$). 1pt

c) Consider: 1pt

$$f(n) = \begin{cases} k & \text{if } n = 2k \\ -k & \text{if } n = 2k+1 \end{cases}$$

The function f is a bijection from $\mathbb{N}$ to $\mathbb{Z}$ (see figure 3).

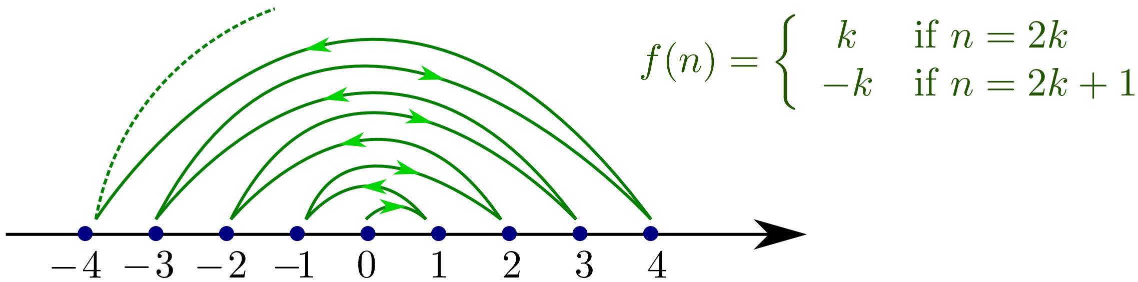

Figure 3: A bijection f between $\mathbb{N}$ and $\mathbb{Z}$. The function f 'jumps' back and forth between right and left.

Ex 9.

a) $f(x) = x^2$, $A = [0, 4]$, $B = [-1, 1]$. Then:

- $f^{-1}(A) = [-2, 2]$, $f^{-1}(B) = [-1, 1]$
- $f^{-1}(A \cap B) = f^{-1}([0, 1) = [-1, 1]$. $f^{-1}(A \cup B) = f^{-1}([-1, 4) = [-2, 2]$. Thus,

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B) \quad \text{and} \quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

b) "$\subset$": Take $x \in g^{-1}(A \cap B)$. There exists $y \in A \cap B$ such that $g(x) = y$. Since $y \in A$, we also have $x \in g^{-1}(A)$. Similarly, since $y \in B$, we have $x \in g^{-1}(B)$. Therefore $x \in g^{-1}(A) \cap g^{-1}(B)$. That proves $g^{-1}(A \cap B) \subset g^{-1}(A) \cap g^{-1}(B)$.

"$\supset$": Now take $x \in g^{-1}(A) \cap g^{-1}(B)$. Thus, $g(x) \in A$ and $g(x) \in B$. Therefore $g(x) \in A \cap B$, in other words $x \in g^{-1}(A \cap B)$. Thus, $g^{-1}(A) \cap g^{-1}(B) \subset\in g^{-1}(A \cap B)$.