MAT 371: Homework 3 (01/29)

1 Chapter 1.6

Ex 3.

(a) The diagonal element x_d is not necessarily a rational number (i.e. $x_d \notin \mathbb{Q}$), neither the element x_* where we add $'+1'$ to each decimal. Thus, there is not necessarily a pre-image to x_* by the function f.

(b) It could be a problem if the diagonal entry x_d would be such a number with two different decimal representations. But that would imply that the decimal expansion of x_d *terminates* (i.e. only $'0'$ or $'9'$ at infinity). That cannot be the case since the diagonal entries are taken from all numbers between $(0,1)$ and there are infinitily many numbers written without 0 and 9.

Ex 4. [3pts]
We proceed by contradiction. Suppose $f : \mathbb{N} \to S$ bijection. We once again consider the table where we list the values of f and consider the diagonal entry (i.e. $x_d = (1, 0, 1, 0, 0, \dots)$). We consider then the element x_* where we flip 0 and 1 (i.e. $x_* = (0, 1, 0, 1, 1, \dots)$). Since f is a bijection, there should exist $n_* \in \mathbb{N}$ such that $f(n_*) = x_*$. Taking the n_* element of this list, we obtain a contradiction $''1 = 0''$.

Ex 9.
Let $I = (0,1)$. Since $\mathbb{R} \sim I$, it is equivalent to show that $I \sim P(\mathbb{N})$. We are going to proceed in two steps. First, we show that $P(\mathbb{N})$ is *larger* than I by construcing a function f injective from I to $P(\mathbb{N})$. Then, we show that I is also *larger* than $P(\mathbb{N})$.

Let $x \in I$ and consider its decimal representation $x = 0.d_1d_2d_3\dots$. We can associate the set:

$$\{d_1,\ 10^2 + d_2,\ 10^3 + d_3, \dots\}.$$

This set is denoted $f(x)$ thus defining a map $f : I \to P(\mathbb{N})$. For instance, $x = .3241 \in I$ gives the set $f(x) = \{3, 12, 104, 1001\} \in P(\mathbb{N})$. It is clear that if $x \neq y$, then the image sets $f(x)$ and $f(y)$ are different, thus f is injective (we also say that f is an *embedding*). Therefore, there is *at least* as many elements in $P(\mathbb{N})$ than in I.

Consider now a set A in $P(\mathbb{N})$, i.e. $A = \{n_1,\ n_2,\ n_3, \dots\}$. We associate a number x_A in I by doing the following:

$$x_A = .\underbrace{11\dots11}_{n_1+1}\underbrace{22\dots22}_{n_2+1}\underbrace{11\dots11}_{n_3+1}\dots$$

For instance, $A = \{0, 3, 7\}$ gives $x_A = 0.1222211111111$. The function $f : A \to x_A$ is also injective and therefore there is *at least* as many elements in I than in $P(\mathbb{N})$.

2 Chapter 2.2

Ex 1. [4pts]

The definition of 'vercongent' implies that there exists $\varepsilon > 0$ such that $|x_n - x| < \varepsilon$ for all n. Thus, any bounded sequence will be 'vercongent': suppose (x_n) bounded by M. Then take $\varepsilon = M + 1$ and $x = 0$, we have that $|x_n - x| < \varepsilon$ for all n.

A sequence that is 'vercongent' can also be divergent: take $x_n = (-1)^n$. It is clearly divergent but i it is also bounded and therefore 'vercongent' to zero. Notice that we can also say that x_n is vercongent to 1 (or any finite number for that matter).

Ex 2. [3pts]

(a) Fix $\varepsilon > 0$. We estimate:

$$\frac{2n+1}{5n+4} - \frac{2}{5} = \frac{10n+5-10n-8}{25n+20} = \frac{-3}{25n+20}.$$

Thus, we want $\left|\frac{3}{25n+20}\right| < \varepsilon$.

Take N such that $N \geq \frac{3}{25\varepsilon}$. For any $n \geq N$, we have: 1pt

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \left|\frac{3}{25n+20}\right| < \frac{3}{25n} \leq \frac{3}{25N} < \varepsilon.$$

(b) Fix $\varepsilon > 0$. We would like to have:

$$\frac{2n^2}{n^3+3} < \varepsilon$$

A sufficient condition would be: $\frac{2n^2}{n^3} < \varepsilon$.

Take N such that $N > \frac{2}{\varepsilon}$. Then for any $n \geq N$, we have: 1pt

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

(c) Fix $\varepsilon > 0$. To have $|\sin(n^2)/n^{1/3}| < \varepsilon$, a sufficient condition is to have $1/n^{1/3} < \varepsilon$.

Take N such that $N > 1/\varepsilon^3$. For any $n \geq N$, we have: 1pt

$$\left|\frac{\sin(n^2)}{n^{1/3}} - 0\right| \leq \frac{1}{n^{1/3}} \leq \frac{1}{N^{1/3}} < \varepsilon.$$

Ex 4.

(a) Take $a_0 = 1$, $a_1 = 0$, $a_2 = 1$,...The sequence does not converge.

(b) Not possible. Suppose $a_n \to a$ with $a \neq 1$. Take $\varepsilon = |a-1|/2 > 0$. Since the sequence converges, there exists N such that

$$|a_n - a| < \varepsilon \quad \text{for } n \geq N. \tag{1}$$

Since there are infinitely many 1, we can also find $n_0 > N$ such that $a_{n_0} = 1$. But then:

$$|a_{n_0} - a| = |1 - a| > \varepsilon.$$

Contradiction with (1).

(c) Take $a_0 = 1$, $a_1 = 0$. Then $a_2 = a_3 = 1$, $a_4 = a_5 = 0$. After that, $a_6 = a_7 = a_8 = 1$, $a_9 = a_{10} = a_{11} = 0$, etc...The sequence is divergent and there are sequence of 1 of any length.

Ex 5.

(a) Let $a_n = [[5/n]]$. For $n > 5$, we have $a_n = 0$. Thus, $a_n \to 0$.

(b) Let $a_n = [[(12+4n)/3n]]$. It is clear that $(12+4n)/3n$ converges to $4/3$. Take $\varepsilon = 1/3$, there exists N such that for $n \geq N$, we have:

$$\frac{4}{3} - \varepsilon < \frac{12+4n}{3n} < \frac{4}{3} + \varepsilon$$

Thus, $\frac{12+4n}{3n}$ is always stricly between 1 and 2 for $n \geq N$. We deduce that $[[\frac{12+4n}{3n}]] = 1$ for $n \geq N$. Thus, $a_n \to 1$.